Life Expectancy, Human Capital, Social Security and Growth

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Abstract

We analyze the effects of changes in the mortality rate upon life expectancy, education, retirement, human capital and growth in the presence of social security. Our starting point is Boucekkine et al. (2002), a continuous time growth, overlapping generations model in which individuals choose the time length of education and retirement age. We introduce a pay-as-you-go social security system in which pensions depend on past contributions. Although this has a positive effect on education, pension benefits also favor reductions in retirement age which, in turn, lowers incentives to education. The net effect is a reduction in both education time and planned retirement.

JEL Classification Numbers: O40, H55, J10

KEY WORDS: Mortality Rate, Social Security, Growth
1 Introduction

In this article we analyze the effects of changes in life expectancy, as a consequence of exogenous falls in the mortality rate, on education time, retirement age, human capital accumulation and economic growth under a pay-as-you-go social security system.

This article intends to integrate two streams of economic literature generally considered separately: i) life expectancy and endogenous growth, and ii) aging, unfunded social security and exogenous economic growth.

• First, many articles published in the recent literature deal with the links between life expectancy and endogenous growth, both from empirical and theoretical points of view.

Regarding empirical works, the hypothesis that reductions in mortality rates have caused higher growth rates is partially supported. By using time series data, Rodriguez and Sachs (1999) find a positive effect of life expectancy upon GDP growth rate in Venezuela in 1970-90. However, Malmberg (1994) finds a negative relationship in Sweden in 1950-89. Analysis of cross section data shows that the relationship between life expectancy and growth is not monotonous either. Preliminary data from Latin America and Caribbean countries show that GDP growth is positively associated with life expectancy. [See World Health Organization (1999), Box 1.2, p. 9.] Barro and Sala-i-Martin (1995), using a sample of 97 countries, estimate that an increment in life expectancy of 13 years would increase per capita growth rate by 1.4% per year.1 Finally, some other studies have found mixed evidence: increases in life expectancy have paralleled higher growth rates for low life

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1Bassanini and Scarpetta (2001) study the effect of an additional year of education upon long run product for 21 OECD countries for 1971-1998: an additional year of education would increase per capita output slightly less than 6%. [See Bassanini and Scarpetta (2001), p. 19, and references therein.]
expectancies, but lower growth rates for high life expectancies. [See Zhang, Zhang and Lee (2003) and references therein.]

A common result in most theoretical studies including some sort of human capital is that an increase in life expectancy increases the period length in which the return to human capital investment is obtained, thus allowing for higher rates of return. This higher return gives rise in turn to higher investment in human capital and, therefore, augmented growth rates. This is the case, among others, of Ehrlich and Lui (1991), Meltzer (1995) and Hu (1999).2

De la Croix and Licandro (1999) build up an economy where the effect of a reduction in the mortality rate upon the duration of education is such that the growth rate becomes higher for high mortality rates (as in underdeveloped countries), but lower for low mortality rates (as in industrialized countries). Zhang and Zhang (2003), Zhang, Zhang and Lee (2001)-(2003) also obtain this result by means of a different channel: not through own education time, but through the expenditure on children’s education. Thus, an inverted U pattern between life expectancy and per capita growth is obtained. The intuition behind the negative sloped part is that average human capital of the labor force becomes more obsolete as life expectancy increases.

Echevarría (2003) suggests an exogenous retirement model in which human capital investment depends positively on the number of remaining active periods until the individual’s retirement. Thus, increases in life expectancy give rise to higher growth rates only if accompanied by simultaneous increments in the active period (i.e., delays in the retirement age).

Boucekkine, De la Croix and Licandro (2002), assuming uncertain lifetime hori-

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2 Kalemli-Ozcan, Ryder and Weil (2000) show that if mortality falls, life expectancy rises, thereby causing higher education levels. Growth rate is unaffected, however, as growth is identically equal to zero in their model.
zon and endogenous retirement age, obtain the inverted $U$ pattern mentioned above once more. Starting from Boucekkine, De la Croix and Licandro (2002), but allowing for physical capital along with human capital in a certain lifetime horizon, Echevarría (2004) obtains the same relationship.

Regarding theoretical works which assume only physical capital, in a model à la Blanchard-Yaari, Reinhart (1999) shows that if mortality drops and life expectancy goes up, then individuals discount the future less, so that savings and physical capital investment increase and so does growth. A similar result can be found in Futagami and Nakajima (2001): greater life expectancy causes an augmented growth rate because the savings rate is higher. Fuster (1999) also obtains the inverted $U$ pattern between life expectancy and growth above mentioned, but through a different mechanism based on accidental bequests.

Second, a considerable number of articles in the recent literature deal with the connections between population aging, pay-as-you-go social security, retirement and exogenous growth. See, among others, De Nardi (1999), Galasso (1999), Huggett (1999), İmrohoroğlu et al. (1999), Kotlikoff (1999). This is not surprising given the concern in industrialized economies about the sustainability of current social security systems, mostly unfunded, and the ongoing aging of their populations. Of course this setup does not allow for an analysis of the consequences on growth. One recurring subject in this literature is the effect of social security upon workers’ voluntary retirement. Along these lines, the available empirical evidence suggests that, at least for the US economy, social security is relevant for retirement issues, even though there is no total agreement on the effects of variations in the generosity of the social security program. [See, among others Diamond and Gruber (1997), Coile and Gruber (2000), Fabel (1994), Kalemli-Ozcan (2002a).]

In this article we attempt to integrate both types of literature, while allowing for
the possibility that the social security system affects not only retirement incentives but also the economy’s growth, that is to say, allowing for endogenous growth. Some previous works partly follow this line of research. In an economy with bequests and endogenous fertility, Zhang and Zhang (2003) show that pay-as-you-go social security in general gives rise to higher growth rates as fertility is reduced and investment in human capital for each child is increased. Along the same lines, assuming unfunded social security, Zhang, Zhang and Lee (2001) show that an increase in life expectancy caused by a fall in the mortality rate of older (retired) individuals causes lower fertility but higher investment in human capital and growth. If social security is funded, however, the result depends on whether bequests are made or not and how the preferences weight the number of children relative to their wellbeing. A common assumption to these works is exogenous retirement.

More precisely, we analyze the effects of changes in uncertain lifetime horizons, caused by exogenous changes in mortality rates, on education periods, human capital accumulation and growth under an unfunded social security system and endogenous retirement. Our starting point is Boucekkine et al. (2002). It is, in essence, an overlapping generations model with uncertain, finite lifetime horizon. Fertility and mortality rates are exogenous, and individuals choose their optimal education time length and retirement, thereby way influencing human capital accumulation and the economy’s growth.

In this model there is endogenous growth because aggregate human capital exerts a positive externality on individual human capital production. The source of growth is the knowledge that individuals accumulate throughout their lifetimes and which is intergenerationally transmitted. The rate at which this knowledge (human

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3Sánchez-Losada (2000), assuming exogenous fertility, also studies the relationship between unfunded social security and growth. He finds that a higher social security contribution rate may increase growth: if there is overaccumulation, savings and physical capital investment fall, but investment on children’s human capital is raised.
capital) is accumulated depends on the time that each generation devotes to pro-
ducing knowledge which, in turn, depends on the average knowledge throughout the
economy. Additionally, we explicitly assume that average human capital depends
on the age structure of population, a common feature of *vintage* models.

In our extension we include an unfunded social security system whose pension
bene… ts depend on the contributions made by workers during their active periods.
According to this design, social security will influence not only *individual* decisions
(namely, years of education and retirement age), but also *aggregate* magnitudes such
as growth. Notice that the return to human capital investment is not constrained to
labor income while active, but also extends to pensions during retirement. Likewise,
voluntary retirement age will depend on the generosity of the public pension system.

In this economy we will see that extended life expectancy at birth implies more
years of education and a delay in voluntary (planned) retirement.

- An increase in the length of time devoted to education may exert an ambiguous
effect on the growth rate: partly positive, as workers would be more skilled, with
higher individual human capitals; and partly negative, because individuals postpone
entry into the labor market, thereby reducing aggregate human capital. These effects
can vary in the presence of social security. First, the implicit depreciation of human
capital investment is reduced: if pension benefits depend on contributions made
throughout workers’ active lives, they can be interpreted as delayed labor income.
This is a positive effect. Second, if, as a result, the number of workers fell, a lower
replacement rate would be required to keep the social security budget balanced (for a
given contribution rate). But the corresponding reduction in pension benefits might
in turn reduce the incentives to education.

- In principle, an increase in the retirement age would have a negative impact
on growth as the average age of human capital would increase. However, the result
might be the opposite, because aggregate human capital would enlarge. If a social security system that allows workers to obtain revenues after retirement is assumed, the incentives to early retirement are obvious.

This article contains two different parts. In part one we solve analytically the individual problem (individuals and firms), the steady state per capita growth rate and the social security budget balance. We characterize the parameter space which determines the type of solution for the individuals’ problem whose existence and uniqueness we prove. Furthermore, we prove the existence and uniqueness of steady state per capita growth rate and social security balance for the case of interior solutions for education and retirement.

In part two, given that the system of equations turns out to be highly non-linear, explicit solutions are excluded. Thus, we limit our results to numerical solutions to comparative statics exercises between steady states. We analyze how our theoretical economy responds to changes in exogenous parameters: the mortality rate falls giving rise to increments in life expectancy and increments in the social security contribution rate. In particular, we look at the responses of education periods, retirement age, dependency ratio, per capita growth rate and social security replacement rate.

We show that the effects of increments in life expectancy upon education periods, retirement age and replacement rate are qualitatively the same whether they are caused by drops in the mortality rate of younger individuals or by drops in the mortality rate of the elder. As for the effects upon the dependency ratio and the per capita growth rate, however, those depend on the behavior of population growth and fertility rates.

Concerning increments in the social security contribution rate, we show that both education periods and retirement age decrease, and the dependency ratio increases. Even though the presence of unfunded social security (whose pension benefits de-
pend positively on the worker’s history wage earnings) may induce higher investment in education, it may also raise incentives to early retirement. If this second effect is strong enough, the net effect upon education might be negative as in our case. Furthermore, when looking at aggregate variables, the replacement rate upon (net) wages remains almost constant, the ratio of (expected and discounted) pension benefits to social security contributions drops and, finally, the per capita growth rate falls.

The rest of the article is organized as follows. Section 2 shows the main demographic features of our model. Readers interested only in the economic aspects may proceed to the next Section. Section 3 introduces the economic model: the individual problem, the aggregate technology of production, the optimal education and retirement age, the aggregates, the social security balance and the balanced growth path. The numerical example and the results are shown in Section 4. Section 5 shows the conclusions. A mathematical Appendix contains the formal proofs.

2 Demographic structure

Individuals face an uncertain lifetime horizon with a positive, age increasing instantaneous mortality rate, so that there exists a maximum attainable age.\footnote{This Section closely follows the first Section in Boucekkine et al. (2002).}

More precisely, the demographic structure in this economy is characterized by the survival rate distribution

\[
m(a) = \frac{\alpha - e^{-\beta a}}{\alpha - 1}, \ 0 \leq a \leq J, \alpha > 1, \beta < 0,
\]

where \( m(a) \) represents the probability of surviving until, at least, the age \( a \). \( J \) denotes the maximum attainable age which corresponds to a 0 survival probability,

\[
m(J) = 0 \iff J = \frac{-\ln \alpha}{\beta} > 0.
\]
One can obtain Yaari-Blanchard’s perpetual youth model with a survival probability
\( m(a) = e^{-\beta a} \) for \( \alpha = 0 \) and \( \beta > 0 \) as a limiting case [Blanchard (1985)]. Population
in this economy is stable in that, in the absence of migratory movements, the age
distributions of fertility and mortality rates have remained constant for a long enough
time. Therefore, the age distribution and the growth rate of population are constant
as well [Schoen (1988)]. We do not consider explicitly the age distribution of the
fertility rate in our model. However, we will show the links between parameters
\( \alpha \) and \( \beta \), the population growth rate and the average fertility rate later on [see
equation (9)].

Increments in both \( \alpha \) and in \( \beta \) imply higher survival probabilities, but in a dif-
ferent manner. An increase in \( \alpha \) means a reduction in young individuals’ mortality;
however, a higher \( \beta \) is equivalent to lower mortality for old individuals.

The probability of surviving at most until the age of \( a \) is given by \( 1 - m(a) \);
therefore, the probability of surviving until (exactly) the age of \( a \) is equal to
\[
 p(a) = \frac{d[1 - m(a)]}{da} = -\frac{\beta e^{-\beta a}}{\alpha - 1} > 0. \tag{3}
\]

Denoting by \( x \) a random variable meaning “remaining lifetime until death”, one
obtains the probability of surviving exactly \( x \) additional years for an individual of
age \( a \) from (1) and (3) as
\[
 p(x/a) = \frac{p(x + a)}{m(a)}, \quad 0 \leq x \leq J - a. \tag{4}
\]

From (4) one obtains the (conditional) life expectancy for an individual of age \( a \),
or the average number of additional periods that one individual of age \( a \) expects to
live as
\[
 EV(a) = E[x/a] = \int_0^{J-a} xp(x/a)dx =
 e^{-\beta a} \left\{ 1 - \frac{[1 + (J - a)\beta] e^{-\beta (J-a)}}{\beta(e^{-\beta a} - \alpha)} \right\}. \tag{5}
\]
From (5) and (2) it is possible to solve $EV(0)$ for two particular cases. The life expectancy at age $a = J$, $EV(J) = 0$; and also at age $a = 0$, i.e., the life expectancy at birth (to which we will refer as, simply, life expectancy),

$$EV(0) = \frac{1}{\beta} - \frac{\alpha \ln \alpha}{(\alpha - 1)\beta} > 0. \quad (6)$$

One can obtain, also as a limiting case for $\alpha = 0$ and $\beta > 0$, Yaari-Blanchard’s perpetual youth model [Blanchard (1985)] with a life expectancy which turns out to be age independent, $EV(a) = 1/\beta$.

$EV(0)$ is increasing both in $\beta$ and in $\alpha$, but (as we mentioned above) increases in $\alpha$ and in $\beta$ give rise to different kinds of increments in life expectancy. Along these lines, Kalemli-Ozcan (2002b), p. 411, claims that during the last two centuries life expectancy at birth has doubled in most parts of the world mostly due to larger falls in child rather than adult mortality. Boucekkine et al. (2002), quoting Kelley and Schmidt (1995), claim that in less developed countries mortality drops concentrate on young and working age individuals. Zhang, Zhang and Lee (2001) claim that during the early stages of mortality falls these concentrate on the younger population (at ages previous to reproduction); but as mortality keeps going down from low enough levels, most additional years are gained at ages after retirement age. From (6) it is easy to show that\footnote{The sign of $\partial EV(0)/\partial \alpha$ is proven by defining $f(\alpha) = \alpha - 1 - \ln \alpha$, and checking that $f(1) = 0$, $f'(1) = 0$ and $f''(\alpha) = \alpha^{-2} > 0$. Recall the restriction $\alpha > 1$.}

$$\frac{\partial EV(0)}{\partial \beta} = -\frac{1}{\beta} EV(0) > 0, \quad \frac{\partial EV(0)}{\partial \alpha} = \frac{-(\alpha - 1 - \ln \alpha)}{\beta(\alpha - 1)^2} > 0.$$ 

Population is assumed to grow at an exogenous, constant rate $n$, so that the measure of births at $\tau$ can be expressed as

$$\pi(\tau) = \zeta e^{n\tau}, \quad \zeta > 0, \quad n \geq 0. \quad (7)$$
Given (1), (2) and (7), after some algebra, one obtains the measure of population at \( \tau \) as the sum of measures of individuals born between \( \tau \) and \( \tau - J \) who have survived until \( \tau \),

\[
P(\tau) = \int_{\tau-J}^{\tau} \pi(t)m(\tau - t)dt = \zeta e^{\eta \tau} \kappa,
\]

where \( \kappa \) is defined as

\[
\kappa = \frac{\alpha \beta (1 - \alpha^{n/\beta}) + n(\alpha - 1)}{(\alpha - 1)(\beta + n)n} > 0.
\]

From (7) and (8) it is easy to interpret \( \kappa \) as the inverse of the fertility ratio, where this is defined as the ratio of the measure of births to the measure of total population at (any) moment \( \tau \), \( 1/\kappa = \pi(\tau)/P(\tau) \). Thus, an increase in life expectancy (regardless of whether \( \alpha \) or \( \beta \) becomes larger) implies two extreme cases: i) a higher population growth rate \( n \) if the fertility ratio is kept constant, or ii) a reduction in the fertility ratio \( \kappa^{-1} \) if the population growth rate is unchanged. 

6 See Kalemli-Ozcan (2002b) who reports the fall in fertility rates for Holland, Sweden, United Kingdom, Africa, Asia and Latin America.

7 See, among others, Kalemli-Ozcan (2002b) as an example of endogenous demographic structure, or Zhang and Zhang (2003) and references therein on effects of social security on fertility rates.

The demographic structure that we use here, characterized by (1) and \( n \) [or, alternatively, by (1) and \( \kappa \)] is exogenous because neither the number of children is an individual choice, nor the survival probability depends, for instance, on per capita income or (private or public) health expenditure.

From (1), (7) and (8) it is possible to obtain the probability density function for age \( a \) as the ratio of the measure of individuals born at \( \tau - a \) and still alive at \( \tau \) to total population at \( \tau \)

\[
f(a) = \frac{\pi(\tau - a)m(a)}{P(\tau)} = \frac{e^{-n\alpha}(e^{-\beta a} - \alpha)}{(1 - \alpha)\kappa}.
\]

The instantaneous mortality rate at age \( a \), denoted as \( \xi(a) \), is defined as the ratio of the measure of individuals who survive exactly until the age \( a \) (i.e., who die at
age $a$) to the measure of individuals surviving at least until the age $a$. Thus, from (1), (3) and (7) one obtains

$$\xi(a) = \frac{-dm(a)/da}{m(a)} = \frac{p(a)}{m(a)} = \frac{-\beta e^{-\beta a}}{\alpha - e^{-\beta a}} > 0,$$

which is strictly increasing in $a$ (recall that we are assuming that $\alpha > 1$ and $\beta < 0$):

$$\frac{\partial \xi(a)}{\partial a} = \frac{\alpha \beta^2 e^{-\beta a}}{(\alpha - e^{-\beta a})^2} > 0.$$  

Once more, we can obtain Yaari-Blanchard’s perpetual youth model with a positive, age independent mortality rate as a limiting case: if $\alpha = 0$ and $\beta > 0$, then $\xi(a) = \beta > 0$. [See Blanchard (1985).] Increasing mortality rate with age conveys a degree of realism to the demographic structure that we are using here. This feature, along with its simplicity [as only three parameters $\alpha$, $\beta$ and $n$, and two equations, (1) and (9), are needed if migratory movements are absent], make this demographic structure highly attractive for theoretical models.

The mean age and the median age of population, $\bar{a}$ and $\hat{a}$, respectively, can be obtained from (2) and (10) as $\bar{a} = \int_0^\infty f(a)da$ and $0.5 = \int_0^\hat{a} f(a)da$, or

$$\bar{a} = \frac{1}{(\alpha - 1)\kappa} \left\{ \frac{\alpha [1 - (1 + nJ)e^{-nJ}]}{n^2} \right. - \frac{1 - [J(\beta + n) + 1] e^{-J(\beta + n)}}{(\beta + n)^2} \left\} , \quad (12)$$

and

$$0.5 = \frac{\alpha(\beta + n) - n}{(\alpha - 1)\kappa(\beta + n)n} + \frac{1}{(\alpha - 1)\kappa} \left[ \frac{e^{-(\beta + n)\hat{a}}}{\beta + n} - \frac{\alpha}{n} e^{-n\hat{a}} \right] . \quad (13)$$

In spite of the simplicity of this demographic model, it is possible to approximate observed age distributions in terms of life expectancy, maximum age and median age reasonably well. [See Table 1]

[INSERT TABLE 1 AROUND HERE]

The pattern is clear: falls in the rate of population growth and in the rate of mortality accompany increments in life expectancy at birth and in the mean age, median age and maximum attainable age.
For illustration purposes only, Figures 1.a – b show how changes in $\alpha$ affect the growth rate $n$, the life expectancy $EV(0)$, the mean age $\bar{a}$ and the median age $\hat{a}$. Falls in the mortality rate through increments in $\alpha$ force population aging: life expectancy, mean age and median age are raised; likewise, the population growth rate $n$ increases. Although not shown for reasons of space, differences about whether the origin is an increase in $\alpha$ or in $\beta$ (or whether $n$ or $\kappa$ is kept constant) are purely quantitative.

3 The economy

Following Boucekkine et al. (2002), time is represented as a continuous variable. At each instant $\tau$ there exists a continuum of cohorts born at different instants $t$. A unique good is produced and it can be consumed, but not accumulated in the form of physical capital. Its price is normalized to one. Production technology uses human capital as the only production factor.

We assume that perfect annuity markets exist. Individuals do not save in physical capital (it does not exist), but in annuities. This kind of asset yields a return to its holder as long as he/she is alive. After his/her death, the property of the asset goes back to the insurance company that issued the asset. Thus, even if there were physical capital and individuals were not altruistic (so that they did not intend to leave bequests to their heirs), they would always prefer to save in annuities rather than in physical capital. The return of annuities would always be higher than that of physical capital because, in exchange, they would give back the annuities to the issuing company in case of death. This way the problem posed by unintended positive bequests is removed. Assuming that negative bequests are forbidden by law, individuals would also prefer to borrow in annuities. In exchange for cancelling
out the debt in case of death, borrowers are forced to paying an extra return that compensates the lending company for the default risk in case of death.\footnote{This type of institution is often used in theoretical models as a means to avoid accidental bequests. For instance, Zhang, Zhang and Lee (2001), Fuster (1999), De la Croix and Licandro (1999), Boucekkine \textit{et al.} (2002). In the following section we show that, first, the instantaneous return of annuities depends on the individual’s age. Regardless of whether he/she is a borrower or a lender, the return is equal to the instantaneous probability of death which depends on age as we have seen in (11). And, second, insurance companies issuing annuities obtain zero profits in equilibrium. [Yaari (1965)].}

\section{The individual problem}

Denoting by $t$ birth date and $\tau$ calendar time, the problem that an individual faces consists in finding the consumption path $C(t, \tau)$, the length of the education period $T(t)$ and the retirement age $R(t)$ which maximize his/her expected lifetime utility.\footnote{In our case retirement does not obey workers’ health related issues, for instance, but leisure time preference. [See Sabatini and Mitchell (1999).]}

Workers’ education has both \textit{micro} consequences (higher labor income) and, as we will see later on, \textit{macro} consequences (eventually, higher economic growth for the whole economy in the aggregate).

Instantaneous utility depends linearly on consumption.\footnote{We first tried to use a \textit{CRRA} (logarithmic, in particular) instantaneous utility function in order to obtain an explicit solution for consumption in a previous version of this article. However, the non linearity of the model increased substantially, giving rise to a multiplicity of solutions.} Disutility from time spent on education and working (\textit{i.e.}, other than in retirement) goes up as time goes by and individual’s age $\tau - t$ increases. It also depends negatively on average human capital in the economy $\bar{H}(t)$ at birth (which, as we will see below, is directly proportional to the individual human capital). Therefore, leisure time is “quality” type \cite[see, \textit{e.g.},][]{MilesiFerrettiRoubini1998} and references therein]. In sum, if devoting time to education and working means less leisure time, lifetime utility depends negatively on retirement age.

Moreover, given that we will consider only steady state paths along which individual choices $R(t)$ and $T(t)$ remain time invariant, the marginal disutility of post-
poning retirement an additional period must be proportional to $H(t)$. This is so because, as we will see, marginal utility out of the additional labor income obtained as a result of postponing retirement one period is also proportional to $H(t)$.

Thus, expected lifetime utility is given by

$$
Z_t + J_t C(t; m(t; \mu)) - 1/\phi \int_t^{t+R(t)} (\tau - t) m(\tau - t) d\tau, \phi > 0 \tag{14}
$$

where $1/\phi$ stands for the disutility which both education and work time represent in terms of lost leisure.

The human capital with which this individual enters the labor market $h(t)$ depends on the number of periods devoted to education $T(t)$ and on the human capital in the economy at the time of his/her birth $H(t)$. In particular,

$$
h(t) = \mu H(t)T(t), \mu > 0 \tag{15}
$$

There is, therefore, an externality in the production of human capital. It seems reasonable to assume that for a given education period, the human capital the individual accumulates is higher the higher the knowledge in the economy as a whole. This is, therefore, a public good that individuals enjoy but do not have to pay for. Similar mechanisms have been used previously in the theoretical literature. [See, e.g., Zhang and Zhang (2003), Zhang, Zhang and Lee (2001)-(2003), Azariadis and Drazen (1990), Lucas (1988), Lucas (1990), Einarsson and Marquis (1996), Echevarría (2003)-(2004) or Nerlove et al. (1993)]. Also, empirical evidence largely supports the positive effect of class and school composition on individual students’ educational attainment or the positive effect of local workers with longer education upon individual wages. [See Benabou (1993) and references therein.]

During his/her active life the individual is paid a gross wage per unit of efficient labor equal to $\omega(\tau)$, and pays a (pay-as-you-go) social security tax at a constant rate $s \in (0, 1)$. Thus, the net labor income obtained by this individual at time $\tau$ is
equal to

\[ w(t, \tau) = (1 - s)\omega(\tau)h(t). \]  

(16)

For simplicity, we assume that there is no depreciation of individual human capital while individuals remain on-the-job. Along these lines, Stokey and Rebelo (1995) claim that the largest source of depreciation of aggregate human capital comes from the fact that lifetimes are finite. Therefore, OLG models allow a more satisfactory treatment of this issue than infinite horizon representative agent models. This, in turn, raises a new problem: how human capital is transmitted from one generation to the next. In our model current generations learn from previous generations: they take advantage of the accumulated knowledge in the society when they are in their education period.

After retirement, the individual is paid a pension benefit equal to \( b(t) \). The relationship between social security contribution and pension benefit is given by the replacement rate \( \theta \) which we define (purely for analytical convenience) in terms of average net wage income obtained during the active period,

\[ b(t) = \theta(1 - s)\bar{w}(t), \]  

(17)

where

\[ \bar{w}(t) = \begin{cases} 
\int_{t+R(t)}^{t+R(t)+T(t)} \frac{h(t)\omega(\tau)}{R(t)-T(t)} d\tau = h(t)\bar{w}(t), & \text{if } R(t) > T(t) \\
0, & \text{if } R(t) = T(t) 
\end{cases} \]  

(18)

denotes the average gross wage income earned along the same period. That is, \( \bar{w}(t) \equiv [R(t) - T(t)]^{-1} \int_{t+T(t)}^{t+R(t)} \omega(\tau)d\tau \) represents the average gross wage per efficiency unit earned while active.\footnote{In some countries pension benefits are linked to the worker’s wage history: that is the case, among others, of the US [Diamond and Gruber (1997)] and Spain [Boldrin et al. (1997)]. In other countries, such as the UK, Holland or Sweden, pension benefits are the universal type. [See Miles (1999).] Zhang and Zhang (2003) assume a mixed setup: part of the pension benefits is based on the wage income obtained during the active period, while the rest is of a universal nature.} \footnote{Pension benefits in our model are proportional to the average wage income earned while active for simplicity, but alternative assumptions could be made. For instance, the relationship between
the individual takes into account that more education time *not only* means higher wages while active, *but also* higher pension benefits while retired.

In short, the problem that the individual born at time $t$ faces can be formally expressed as

$$\max_{\{C(t, \tau)\}_{\tau=t,T(t), R(t)}} \int_t^{t+J} C(t, \tau) m(\tau - t) d\tau - \frac{\bar{H}(t)}{\phi} \int_t^{t+R(t)} (\tau - t) m(\tau - t) d\tau$$

subject to

$$\begin{cases}
\int_t^{t+J} D(t, \tau) C(t, \tau) d\tau = \int_t^{t+R(t)} D(t, \tau) w(t, \tau) d\tau + \int_{t+R(t)}^{t+J} D(t, \tau) b(t) d\tau, \\
w(t, \tau) = (1-s)\omega(\tau) h(t), \\
h(t) = \mu \bar{H}(t) T(t), \\
b(t) = \theta (1-s) \bar{w}(t), \\
\bar{w}(t) = h(t) \bar{\omega}(t), \\
R(t) \leq J,
\end{cases}\quad(20)$$

where $D(t, \tau)$ denotes the discount factor that applies between $t$ and $\tau$; that is, the price that an individual pays in $t$ for one unit of consumption at time $\tau$ (contingent on being alive at that time).

Upon substituting the second to fourth restrictions into the first one in (20), one obtains the following Lagrangian

$$\mathcal{L} = \int_t^{t+J} C(t, \tau) m(\tau - t) d\tau - \frac{\bar{H}(t)}{\phi} \int_t^{t+R(t)} (\tau - t) m(\tau - t) d\tau - \lambda(t) \left\{ \int_t^{t+J} D(t, \tau) C(t, \tau) d\tau - \int_t^{t+R(t)} D(t, \tau) (1-s)\omega(\tau) \mu \bar{H}(t) T(t) d\tau - \int_{t+R(t)}^{t+J} D(t, \tau) \theta (1-s) \bar{\omega}(t) \mu \bar{H}(t) T(t) d\tau \right\} - v(t) [R(t) - J].$$

$\lambda(t) \geq 0$ denotes the Lagrange multiplier associated with the intertemporal budget constraint (the marginal utility of income), and $v(t) \geq 0$ is the Lagrange multiplier associated with the restriction that retirement age cannot exceed the maximum age. 

Pension benefits and average wage income in the US and in Spain is increasing, of course, but *concave*. This might increase the incentives to early retirement [See Diamond and Gruber (1997), pp. 7 y 8, and Jiménez-Martín and Sánchez (1999) pp. 49 and 50]
limit $J$. Notice that (21) is linear in $C(t, \tau)$, so that we are implicitly assuming that consumption is non-negative.

The corresponding first order necessary conditions are given by

$$\frac{\partial L}{\partial C(t, \tau)} = 0 \iff m(\tau - t) = \lambda(t)D(t, \tau),$$

$$\frac{\partial L}{\partial R(t)} = 0 \iff \frac{H(t)}{\phi} R(t)m[R(t)] = +\lambda(t)\mu\bar{H}(t)T(t)\omega[t + R(t)](1 - s)D[t + R(t), t] -$$

$$-\lambda(t)\theta(1 - s)\bar{\omega}(t)\mu\bar{H}(t)T(t)D[t + R(t), t] - v(t),$$

$$R(t) \leq J, \ v(t)[R(t) - J] = 0, \ v(t) \geq 0 \quad (24)$$

$$\frac{\partial L}{\partial T(t)} = 0 \iff 0 = \lambda(t) \int_{t + R(t)}^{t + R(t)} D(t, \tau)(1 - s)\omega(\tau)\mu\bar{H}(t)d\tau -$$

$$-\lambda(t)\mu\bar{H}(t)T(t)\omega[t + T(t)](1 - s)D[t + T(t), t] +$$

$$+\lambda(t) \int_{t + R(t)}^{t + J} \theta(1 - s)\bar{\omega}(t)\mu\bar{H}(t)D(t, \tau)d\tau.$$

In (23) we obtain that the marginal disutility of postponing retirement for one additional period (in terms of lost leisure) must be equal to the marginal utility out of the augmented consumption that the additional income allows. Suppose that the retirement age is postponed one additional period. Note that $i$) $\lambda(t)$ represents the expected marginal utility out of income; $ii$) the sum of the terms that multiply $\lambda(t)$ is the marginal increase of the discounted future labor income; and $iii$) $v(t)$ is the expected marginal utility out of increasing the maximum lifetime horizon $J$ (relevant if the restriction $R(t) \leq J$ is binding).

By definition, one has that $D(t, t) \equiv 1$ and that $m(t - t) = m(0) = 1$; therefore, from (22) we obtain

$$\lambda(t) = 1, \text{ and } m(\tau - t) = D(t, \tau). \quad (26)$$
Given that the utility function is linear in $C(t, \tau)$, so is the Lagrangian (21): if no restrictions are imposed on the optimal consumption plan, the maximum of (21) is not well defined unless $m(\tau - t) = D(t, \tau)$. That is, budget constraint and indifference curve coincide. If one imposed the non-negativity of $C(t, \tau)$ in an explicit manner, then (21) should be rewritten allowing for slackness variables. In that case, the equality between the discount factor $D(t, \tau)$ and the survival probability $m(\tau - t)$ would be obtained only for interior solutions.

Denoting the instantaneous return of an annuity by $r(x)$, from (1) and (26) one obtains

$$e^{-\int_t^\tau r(x)dx} = \frac{\alpha - e^{-\beta(\tau-t)}}{\alpha - 1}. \quad (27)$$

Upon differentiating both sides of (27) with respect to $\tau$, we obtain that

$$r(\tau) = \frac{-\beta e^{-\beta(\tau-t)}}{\alpha - e^{-\beta(\tau-t)}} \equiv \xi(\tau - t). \quad (28)$$

In other words, the instantaneous rate of return at time $\tau$ for an individual born at $t$ is identical to the instantaneous mortality rate of an individual of age $\tau - t$ defined in (11). An implication of (28) is that, assuming that insurance companies issuing annuities are risk neutral and perfectly competitive, they obtain zero profits. Denoting the stock of assets at time $\tau$ of an individual born at $t$ by $W(t, \tau)$, the costs of the insurance company would be equal to $r(\tau)W(t, \tau)$. But its revenues would be equal to $\xi(\tau - t)W(t, \tau)$ because a fraction $\xi(\tau - t)$ of individuals of age $\tau - t$ would give back all their assets to the company on dying.

From (22) and (26) we obtain that (23) can be rewritten as

$$-\frac{\bar{H}(t)}{\phi} R(t)m[R(t)] + \mu \bar{H}(t)T(t)\omega[t + R(t)](1-s)m[R(t)] -$$

$$-\theta(1-s)\omega(t)\mu \bar{H}(t)T(t)m[R(t)] - v(t) = 0. \quad (29)$$

There are two open possibilities:
i) If the optimal \( R(t) \) is an \textit{interior} solution, \( R(t) < J \), then \( v(t) = 0 \). From (29) one obtains

\[
R(t) = \phi \mu T(t)(1 - s) \{ \omega[t + R(t)] - \theta \bar{\omega}(t) \}. \tag{30}
\]

ii) If the optimal \( R(t) \) is a \textit{corner} solution, then \( v(t) \geq 0 \) and

\[
R(t) = J. \tag{31}
\]

We will show later on that, given the technology for aggregate production, the wage per efficiency unit is constant \( \omega(\tau) = \omega \), so that \( \omega(\tau) = \bar{\omega}(t) = \omega \). Thus, we will have that

\[
R(t) = \min \{ \phi \mu T(t)(1 - s)(1 - \theta) \omega, J \}. \tag{32}
\]

For the same reason, given (26) and (32), (25) can be rewritten as

\[
T(t)m[T(t)] = \int_{t+T(t)}^{t+\min\{\phi \mu T(t)(1-s)(1-\theta)\omega, J\}} m(\tau - t)d\tau + \int_{t+\min\{\phi \mu T(t)(1-s)(1-\theta)\omega, J\}}^{t+J} \theta m(\tau - t)d\tau. \tag{33}
\]

Notice that (33) implies that \( T(t) = T \) and, therefore, from (32) we have that \( R(t) = R \). That is, \textit{optimal education time length and retirement age are constant}.

A key parameter for our discussion and one we will use repeatedly is \( \eta \equiv \phi \mu \). If we define \( \eta_0 \equiv 1/[(1-s)(1-\theta)\omega] \), the following two equations characterize \( T \) and \( R \)

\[
R = \min \left\{ \frac{\eta}{\eta_0} T, J \right\}, \tag{34}
\]

and

\[
Tm(T) = \int_T^{\min\left\{ \frac{\eta}{\eta_0} T, J \right\}} m(\tau)d\tau + \int_{\min\left\{ \frac{\eta}{\eta_0} T, J \right\}}^{J} \theta m(\tau)d\tau. \tag{35}
\]

### 3.2 Technology of aggregate production

We assume that production technology is linear in human capital,

\[
Y(\tau) = \omega H(\tau), \ \omega > 0, \tag{36}
\]
where $Y(\tau)$ denotes aggregate production and $H(\tau)$ aggregate human capital at $\tau$. The latter is equal to the sum of individual stocks of human capital across workers of different ages (born at different $t$’s, but active at $\tau$). This is, in sum, a vintage model as explained in detail in subsection 3.4. Therefore not only is the time $\tau$ at which human capital is measured relevant, but so is the education length $T$, retirement age $R$ and age distribution of the population. Marginal productivity of human capital $\omega$ is constant and equals the (gross) wage per unit of efficiency. The parallelism with $AK$ technologies in which production is proportional to the stock of aggregate capital in equilibrium is obvious. [See, among others, Barro and Sala-i-Martín (1995) for details.]

**3.3 School and retirement in equilibrium**

In this subsection we characterize optimal education periods and retirement age.

We will make the following distinction:

- $a)$ **interior** solutions: $0 < T < R < J$,
- $b)$ **corner** solutions which, in turn, can be of two types:
  - $b.1)$ $0 < T < R = J$. In this case, planned retirement is given by the maximum lifetime horizon, and education period is equal to the upper bound for interior solutions for $T$ (which we will characterize later on); and
  - $b.2)$ $0 = T = R < J$. In this case individuals choose neither to invest in human capital nor to enter the labor market. If so, both labor income and pension benefits are zero. This is possible given our assumption of linear utility from consumption.

**3.3.1 Interior solution:** $0 < T < R < J$.

In this subsection we find the conditions upon parameter $\eta$ for the existence and uniqueness of an interior solution. If the solution is interior, from (34) we obtain
that

\[ R = \frac{\eta}{\eta_0} T, \]

where \( R \) and \( \eta_0 \) (or, equivalently, \( \eta, s, \theta \) and \( \omega \)) must satisfy for \( R > T > 0 \) is that

\[ \eta > \eta_0. \]

Intuitively, given the definitions of \( \eta \) and \( \eta_0 \), for individuals to devote a part of their lifetimes to education and a part to active work one needs: (i) high gross wages (high \( \omega \)), (ii) low social security contribution rates (low \( s \)), (iii) high productivity of investment in human capital (high \( \mu \)), (iv) low disutility of time not devoted to leisure (high \( \phi \)), and (v) low replacement rates (low \( \theta \)).

Note that if the solution is interior, \( R \) is proportional to \( T \) and, in particular, \( R = \eta(1 - \theta)(1 - s)\omega T \). Therefore, for a given \( \theta \), the discouraging effect that a higher \( s \) has upon \( T \) is enlarged when we look at the effect upon \( R \). This point will be relevant when we carry out our numerical exercise in Section 4.

Assuming that \( \eta > \eta_0 \), from (34), (35) and (37), optimal \( T \) is given by

\[ Tm(T) = G_1(T) + G_2(T), \]

where we have defined

\[ G_1(T) \equiv \int_{\frac{T}{\eta_0}}^{\omega T} m(\tau)d\tau, \text{ and } G_2(T) \equiv \theta \int_{\frac{T}{\eta_0}}^{J} m(\tau)d\tau. \]

In other words, for \( T \) to be optimal the cost of an additional education period (in terms of foregone wages) must be equal to the increment in the sum of discounted future wages plus pension benefits as a result of that additional learning period.

Besides, following the case of interior solution so that \( R < J \), (37) implies that

\[ T < \frac{J\eta_0}{\eta} \equiv T_{\text{max}}(\eta) < J. \]
There is, therefore, an upper bound for the optimal $T$ (not only for interior solutions, but also for corner solutions in which $R = J$ and $T \equiv T_{\text{max}}$).

Introducing unfunded social security with positive $\theta$ and $s$, raises the lower bound of $\eta$ for interior solutions slightly higher than the one required in Boucekkine et al. (2002). [See Boucekkine et al. (2002), Lemma 2.3, p. 350, who obtain $\eta > 2$ as a necessary condition for interior solution]. Assuming [as Boucekkine et al. (2002)] $\omega = 1$, one has that $\eta_0 = [(1 - s)(1 - \theta)]^{-1}$, need not be higher than 2. For instance, the productivity of education time in the production of individual human capital need not be so high as to induce individuals to spend a fraction of their lifetimes on accumulating knowledge. Why? Because in the presence of a social security system like ours (unfunded and whose pension benefits depend positively on earned labor income in the past) pension benefits represent an additional incentive to the wage income obtained during the active period. In other words, pension benefits reduce the depreciation of human capital that a finite active period represents for workers.

A graphical argument may be useful to understand the last discussion. [See Figure 2.]

The area $Tm(T)$ must be equal to the sum of areas $G_1(T) \equiv \int_{T_0}^{\frac{\eta T}{\eta_0}} m(\tau)d\tau$ plus $G_2(T) \equiv \theta \int_{\frac{T_0}{\eta_0}}^{J} m(\tau)d\tau$. Therefore, $G_1(T)$ need not be so high as when there is no social security, the optimal $T$ condition being given in that case by $Tm(T) = G_1(T)$.$^{13}$

---

$^{13}$If there were no social security, and given that $m(x)$ is strictly decreasing, one would have in Figure 5 that

$$\left(\frac{\eta}{\eta_0} - 1\right) Tm(\frac{\eta T}{\eta_0}) < \int_{T_0}^{\frac{\eta T}{\eta_0}} m(x)dx < \left(\frac{\eta}{\eta_0} - 1\right) Tm(T).$$

If the solution for $T$ were interior, from the second of the two previous inequalities and (39) one
To analyze the existence and uniqueness of the interior solution, we first define the following auxiliary continuous function in $x$ and $\eta$:

$$M(x, \eta) \equiv (\alpha - 1) [xm(x) - G_1(x) - G_2(x)] =$$

$$= x(\alpha - e^{-\beta x}) + \alpha x \left[ 1 - \frac{\eta}{\eta_0} \right] + \frac{e^{-\beta x} - e^{-\beta x} \frac{\eta}{\eta_0}}{\beta} +$$

$$+ \theta \alpha \left[ \frac{\eta}{\eta_0} x - J \right] + \theta \left[ \frac{e^{-\beta x} \frac{\eta}{\eta_0} - e^{-\beta x}}{\beta} \right].$$

From (39), (40) and (42) one has that $T$ is an interior solution if and only if $x = T$ is a root of equation (42). Therefore, we will be able to discuss the existence and uniqueness of the interior solution upon studying the properties of $M(x, \eta)$.

Our strategy will be as follows:

1) first, we will prove that $M(x, \eta)$ is negative in the origin $x = 0$ and positive at $x = T_{\text{max}} > 0$ for an interval of values of $\eta$: the continuity of $M(x, \eta)$ will assure us that there exists at least one $x \in (0, T_{\text{max}}(\eta))$ for which $M(x, \eta) = 0$.

2) second, we will show that such an $x$ is unique.

The argument is shown graphically in Figure 3.

[INSERT FIGURE 3 AROUND HERE]

The following Proposition gives us sufficient conditions for $M(x, \eta)$ to be negative at $x = 0$.

**Proposition 1** Assume $\beta < 0$ and $\alpha > 1$: if $\theta > 0$, then $M(0, \eta) < 0$.

would get

$$\int_{T}^{t_{\eta_0}} m(x) dx < \left( \frac{\eta}{\eta_0} - 1 \right) \left[ \int_{T}^{t_{\eta_0}} m(x) dx + \theta \int_{t_{\eta_0}}^{J} m(x) dx \right] \Rightarrow$$

$$\Rightarrow \eta > \left[ \frac{2 \int_{T}^{t_{\eta_0}} m(x) dx + \theta \int_{t_{\eta_0}}^{J} m(x) dx}{\int_{T}^{t_{\eta_0}} m(x) dx + \theta \int_{t_{\eta_0}}^{J} m(x) dx} \right] \eta_0.$$

If we assume that $\theta = 0$, $\omega = 1$ and $s = 0$, we will have that $\eta > 2\eta_0 = 2$, the same condition that Boucekkine et al. (2002) obtain [See Boucekkine et al. (2002), Proposition 2.1, p. 350.]
Proof: See Appendix.

The next step consists in obtaining the conditions on \( \eta \) which guarantee that \( M(T_{\text{max}}(\eta), \eta) > 0 \). To this end we define the following auxiliary function \( K(\eta) \equiv M(T_{\text{max}}(\eta), \eta) \). From (2), (41) and (42) it can be shown that

\[
K(\eta) = \frac{1}{\beta} \left\{ \frac{\eta_0}{\eta} \left( \frac{\alpha}{\eta} - 2\alpha \right) \ln \alpha + \alpha (\ln \alpha - 1) + \frac{\alpha}{\eta} \right\}, \tag{43}
\]

which is continuous for \( \eta > 0 \). We next characterize function \( K(\eta) \). We will show in Lemmas 2-8 and Corollary 9 that \( K(\eta) \) is positively sloped for low \( \eta \)'s, that it becomes zero at \( \eta = \eta_0 \), and once it attains a positive unique local (global) maximum for some \( \eta = \hat{\eta} \), it diminishes monotonically and becomes zero at some (unique) \( \eta = \eta^* \). The final result is summarized in Corollary 10.

**Lemma 2.** Assume \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \): if \( \eta < \eta_0 \), then \( K(\eta) \) is strictly increasing.

**Proof:** See Appendix.

The following Lemma says that \( K(\eta) \) equals zero and that it is increasing at \( \eta = \eta_0 \).

**Lemma 3** Assume \( \alpha > 1, \beta < 0 \) and \( \eta_0 > 0 \). i) \( K(\eta_0) = 0 \). ii) If \( \eta_0 > 0 \), then \( K'(\eta_0) > 0 \).

**Proof:** See Appendix.

**Lemma 4** If \( \alpha > 1, \beta < 0 \) and \( \eta_0 > 0 \), then \( K(\eta) \) is negative for all \( \eta < \eta_0 \).

**Proof:** See Appendix.

The following Lemma says that at \( \eta = 2\eta_0 > 0 \) the function \( K(\eta) \) is positive and decreasing.

**Lemma 5** If \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \), then i) \( K(2\eta_0) > 0 \), and ii) \( K'(2\eta_0) < 0 \).
Proof: See Appendix.

The following Lemma proves that there is one unique \( \hat{\eta} \) greater than \( \eta_0 \) and less than \( 2\eta_0 \) such that \( K(\eta) \) attains a local maximum at \( \eta = \hat{\eta} \).

**Lemma 6** If \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \), then one unique \( \hat{\eta} \in (\eta_0, 2\eta_0) \) exists such that \( K'(\hat{\eta}) = 0 \).

*Proof:* See Appendix.

The following Lemma proves that \( K(\eta) \) is strictly decreasing for all \( \eta \) greater than \( 2\eta_0 \).

**Lemma 7** Assume that \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \): if \( \eta > 2\eta_0 \), then \( K(\eta) \) is strictly decreasing.

*Proof:* See Appendix.

The following Lemma proves that the limit of \( K(\eta) \) as \( \eta \) tends to plus infinity is negative.

**Lemma 8** If \( \beta < 0 \) and \( \alpha > 1 \), then \( \lim_{\eta \to \infty} K(\eta) < 0 \).

*Proof:* See Appendix.

The following Lemma proves the existence of one unique \( \eta^* \) greater than \( 2\eta_0 \) such that the function \( K(\eta) \) is equal to zero.

**Lemma 9** Assume \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \). Then i) there is a unique \( \eta^* > 2\eta_0 \) such that \( K(\eta^*) = 0 \), and ii) \( K(\eta) < 0 \) for all \( \eta > \eta^* \).

*Proof:* See Appendix.

This way we have characterized the interval for \( \eta \) such that \( K(\eta) \) takes on strictly positive values. The following Corollary summarizes the properties of \( K(\eta) \).

**Corollary 10** Assume \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \): then i) \( K(\eta) \) is continuous; ii) if
η < η₀, then K(η) < 0; iii) if η = η₀, then K(η) = 0; iv) K(η) > 0 if and only if η ∈ (η₀, η*); v) if η = η*, then K(η) = 0; and vi) if η > η*, then K(η) < 0.

Proof: See Appendix.

The function K(η) is represented in Figure 4.

[INSERT FIGURE 4 AROUND HERE]

The following Proposition gives us the interval for η such that there exists at least one interior solution for T and R satisfying 0 < T < R < J, T < T_{max}, and equations (37) and (39).

**Proposition 11** Existence. Sufficiency. Assume β < 0, α > 1 and η₀ > 0. If η₀ < η < η*, then there is at least one interior solution for T and R which satisfies (37) and (39), and for which 0 < T < R < J, T < T_{max}.

Proof: See Appendix.

The following Proposition gives us sufficient conditions for the uniqueness of an interior solution, i.e., for equation (42) to have a unique root.

**Proposition 12** Uniqueness. Sufficiency. Assume β < 0, α > 1, θ > 0 and η₀ > 0. If η₀ < η < η*, then there is a unique interior solution for T and R which satisfies (37) and (39), and such that 0 < T < R < J, T < T_{max}.¹⁴ ¹⁵

Proof: See Appendix.

The following Proposition gives us necessary conditions for the uniqueness of the interior solution, i.e., for equation (42) to have a unique root.

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¹⁴We owe the the last part of proof to Águeda Madoz, our research assistant.

¹⁵We believe that the proofs of Lemma 2.2 and Lemma 2.3 in Boucekkine et al. (2002) are wrong. The authors claim that “Trivially, lim_{x→+∞} M(x) = +∞ ...”, but this is not true. [See proof of Lemma 2.2 on page 367 and proof of Lemma 2.3 on page 368.] Notice that m(x) as defined in (1) is identically equal to 0 for x ≥ J.
**Proposition 13** Uniqueness. Necessity. Assume $\beta < 0$, $\alpha > 1$, $\theta > 0$ and $\eta_0 > 0$: if the unique solution for $T$ and $R$ is interior, that is to say, $0 < T < R < J$, $T < T_{\text{max}}$, and satisfies (37) and (39), then it must be the case that $\eta_0 < \eta < \eta^*$.

*Proof:* See Appendix. ■

In the next subsection we study corner solutions.

### 3.3.2 Corner solutions: $0 < T < R = J$ and $0 = T = R < J$.

We obtain two possible corner solutions by considering four cases depending on the value of $\eta$: i) $\eta = \eta^*$, ii) $\eta > \eta^*$, iii) $\eta = \eta_0$, and iv) $0 < \eta < \eta_0$. In the third case we will discuss the compatibility of the optimal solution from an individual standpoint with the equilibrium for the whole economy.

- **Case 1:** $\eta = \eta^*$

  Let us assume that $\eta = \eta^*$: in this case, given (37), $T_{\text{max}}(\eta)$ defined in (41), (39), $M(x, \eta)$ defined in (42), $K(\eta)$ in (43) and $\eta^*$ in Corollary 9, $T = T_{\text{max}}(\eta^*)$ and $R = J$ satisfy the condition for an interior solution

  $$T_{\text{max}}(\eta^*)m[T_{\text{max}}(\eta^*)] = \int_{T_{\text{max}}(\eta^*)}^J m(\tau)d\tau + \theta \int_{T_{\text{max}}(\eta^*)}^J m(\tau)d\tau = (44)$$

  Therefore, interior and corner solutions coincide. Notice that by definition of $\eta^*$ in Corollary 9, $\eta^* > \eta_0$ and, therefore, $T_{\text{max}}(\eta^*) \equiv \frac{J\eta_0}{\eta^*} < J$. Moreover, $T_{\text{max}}(\eta^*)$ is the unique solution which, being less than $J$, satisfies (44). The following Proposition states this result.

**Proposition 14** Assume that $\beta < 0$, $\alpha > 1$ and $\eta_0 > 0$: if $\eta = \eta^*$, then $T = T_{\text{max}}(\eta^*) \equiv \frac{J\eta_0}{\eta^*}$ and $R = J$ is the unique solution to (37) and (39) such that $0 < T < J$. 

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\textit{Proof}: See Appendix. \hfill\Box

\textbf{Case 2:} \( \eta > \eta^* \)

The following Proposition states the solution for optimal \( T \) and \( R \) for values of \( \eta \) greater than \( \eta^* \).

\textbf{Proposition 15} Assume that \( \beta < 0, \alpha > 1 \) and \( \eta_0 > 0 \): if \( \eta > \eta^* \), then \( T = T_{\text{max}}(\eta^*) = \frac{J_{\eta_0}}{\eta^*} \) and \( R = J \) is the unique solution to (34) and (35) such that \( 0 < T < J \).

\textit{Proof}: See Appendix. \hfill\Box

Note that \( T = R = J \) is also a solution both in Case 1 and in Case 2. However, by using an indirect utility argument, this solution is dominated by \( T = T_{\text{max}}(\eta^*) = \frac{J_{\eta_0}}{\eta^*} \) and \( R = J \). Leisure is zero in both cases \( (R = J) \) and so are pension benefits. But labor income is zero, and so is consumption, if \( T = R \), while labor income and consumption are positive if \( T = T_{\text{max}}(\eta^*) < R \).

\textbf{Case 3:} \( \eta = \eta_0 \)

Consider now the case of \( \eta = \eta_0 \). From (34) and (35) we obtain

\[ R = \min \{T, J\}, \text{ and} \]
\[ Tm(T) = \int_T^{\min\{T, J\}} m(\tau)d\tau + \int_{\min\{T, J\}}^J \theta m(\tau)d\tau. \]

Given that individuals never survive the age \( J \), it must be the case that \( T \leq J \) and, therefore, \( R = T \) always. This implies, in turn, that individuals never contribute to the social security and, therefore, \( \theta \) must be 0 if the social security budget is balanced.

\( a \) If this budget balance condition is imposed, then optimal \( T \) is given by equation (35) and, therefore, \( Tm(T) = 0 \) which admits two solutions: \( 0 < T = R = J \),
where $T = T_{\text{max}}(\eta_0)$, and $0 = T = R < J$. The latter is preferred to the former. Why? Consumption is zero in both cases: both labor income and pension benefits are zero because $T = R$ [recall equations (17) and (18)]. But leisure is positive ($R < J$) in the latter case, and zero in the former ($R = J$).

b) Suppose, alternatively, that the social security budget balance is not imposed and a $\theta > 0$ is allowed. If so, optimal $T$ is characterized by condition (35) and, therefore, $Tm(T) = \theta \int_T^J m(\tau)d\tau$, which admits two solutions: $0 < T = R = J$, and $0 < T(\theta) = R < J$. In this case, the latter is preferred to the former. If $T(\theta) = R < J$, both labor income and pension benefits are zero (because $T = R$), so that consumption is zero too; but leisure is positive ($R < J$). Whereas if $T = R = J$, consumption is also zero as before, but this time leisure is zero too ($R = J$).

The following Proposition formalizes this result.

**Proposition 16** Assume that $\alpha > 1$, $\beta < 0$ and $\eta = \eta_0 > 0$. i) If social security budget is balanced, then $\theta = 0$, $0 = T = R < J$; ii) if social security budget is allowed to be unbalanced and $\theta > 0$, then $0 < T = T(\theta) = R < J$, where $T(\theta)$ is the solution to $T(\theta)m[T(\theta)] = \theta \int_T^J m(\tau)d\tau$.

*Proof*: See Appendix. ■

**Case 4**: $0 < \eta < \eta_0$

Suppose, finally, that $0 < \eta < \eta_0$. If this is the case, from (34) one has that

i) either $R = J \leq T\eta/\eta_0$, so that $J \leq T\eta/\eta_0 < T$; but this cannot be a solution, as it is meaningless: individuals would study after retirement and death; or

ii) $R = T\eta/\eta_0 \leq J$ which makes sense only if $R = T = 0$. Otherwise, if $T > 0$, one has that $R = T\eta/\eta_0 < T$ and, therefore, $R < T$, which makes no sense. Individuals would retire before completing the education period and entering the labor market.

The following Proposition states this result. The economic meaning of Case 4 (as
that of Case 3) is absent: there would be no human capital, nor production, nor consumption.

**Proposition 17** Assume that $\alpha > 1$ and $\beta < 0$. If $0 < \eta < \eta_0$, then $0 = T = R < J$.

*Proof:* See Appendix.

Table 2 summarizes the results in Propositions 12-17. Note that for $\eta = \eta_0$ we have only considered the possibility of social security budget balance.

[INSERT TABLE 2 AROUND HERE]

### 3.4 Aggregates

In this subsection we obtain the aggregates for consumption and human capital at time $\tau$. To this end we weight individuals’ decisions by the size of the surviving population in the living cohorts, and sum them up across birth dates.

- **Aggregate consumption.** From (1) and (7) aggregate consumption can be expressed as,
  \[ C(\tau) = \int_{\tau-J}^{\tau} C(t, \tau)\zeta e^{nt}m(\tau - t)dt, \]  
  where $C(t, \tau)$ represents consumption at $\tau$ of an individual born at $t$ and $\zeta e^{nt}m(\tau - t)$ denotes the measure of population of $t$-th generation still alive at time $\tau$.

- **Aggregate human capital.** From (1) and (7) we obtain aggregate human capital as
  \[ H(\tau) = \int_{\tau-R(\tau)}^{\tau-T(\tau)} h(t)\zeta e^{nt}m(\tau - t)dt, \]  
  where the last generation to enter the labor market was born at $\tau - T(\tau)$, and the last generation to retire from their jobs was born at $\tau - R(\tau)$. The cohort born at $t$ and still in the labor force has a measure equal to $\zeta e^{nt}m(\tau - t)$, and their members have a stock of individual human capital $h(t)$, making this a *vintage* model.
Vintage models are often used both in economies with physical capital and in economies with human capital. In the first case the aggregate stock of capital installed in firms consists of capital goods of different ages, usually embedding different technologies (with more productive technologies in the more recent ones). In the second (i.e., our) case, younger workers incorporate higher levels of human capital than their predecessors in a growing economy (although without their labor expertise which, for simplicity, we are not considering here).\[16\]

From (46) we define average human capital at time \( \tau \), first introduced in the individual human capital production function (15), as

\[
\bar{H}(\tau) = \frac{H(\tau)}{\kappa e^{\eta \tau}},
\]

(47)

where the denominator represents total population at time \( \tau \), defined in (8).

An indicator that we could use a priori to explain growth in vintage models is the quality of capital, proxied by its average age. In our case, the average age of human capital is given by the average age of active workers \( L \), which given the age distribution (10) is equal to

\[
L(\tau) = \frac{\int_{T(\tau)}^{R(\tau)} a e^{-na(e^{-\beta} - \alpha)} da}{\int_{T(\tau)}^{R(\tau)} e^{-na(e^{-\beta} - \alpha)} (1 - \alpha) da}.
\]

(48)

The numerical exercises in Section 4, however, will not show a monotonic relationship between \( L \) and \( \gamma \).

### 3.5 Social security

Assuming that social security balances its budget on a period by period basis, the following equality must hold at each time \( \tau \)

\[
\int_{\tau-R(\tau)}^{\tau-T(\tau)} s(\tau)h(t)\pi(t)m(\tau - t)dt = \int_{\tau-J}^{\tau-R(\tau)} b(t)\pi(t)m(\tau - t)dt.
\]

(49)

---

\[16\] Among the first type one could mention, e.g.: Gittleman et al. (2003) and Jensen et al. (2001). And among the second type, one could mention Boucekkine et al. (2002), Echevarría (2003)-(2004), Violante (2002), and Neuman and Weiss (1995).
The left-hand-side represents the social security tax revenue from active generations \([i.e., \text{born after } \tau - R(\tau), \text{but before } \tau - T(\tau)]\). The right-hand-side equals the pension benefits paid to retirees \([i.e., \text{individuals born after } \tau - J, \text{but before } \tau - R(\tau)]\).

3.6 Equilibrium

**Definition 18** An equilibrium path for this economy is defined as a sequence of quantities \(\{T(\tau), R(\tau), C(\tau), h(\tau), H(\tau), \tilde{H}(\tau), Y(\tau)\}_{\tau=0}^{\infty}\) and prices \(\{\omega(\tau)\}_{\tau=0}^{\infty}\) such that

i) consumers maximize utility taking the sequences of human capital in the economy and wage per efficiency unit, and the parameters representing the social security policy \(\{s, \theta\}\) as given;

ii) firms maximize profits taking the sequence of wage per efficiency unit as given;

iii) the government chooses the replacement rate \(\theta\), for a given social security tax rate \(s\) such that social security budget is balanced at each instant; and

iv) goods market clears.

In this article we only consider balanced growth paths characterized by the fact that aggregate variables \(\{C(\tau), H(\tau), Y(\tau)\}\) grow at a constant rate or, equivalently, per capita variables \(\{h(\tau), \tilde{H}(\tau), w(\tau)\}\) grow at a constant rate \(\gamma\). Moreover, variables indicating education time duration \(T\) and retirement age \(R\) are constant and, therefore, do not depend on the worker’s birth date.

3.7 The balanced growth path

**Definition 19** Balanced growth paths in this economy are defined as sequences of quantities \(\{T, R, C(\tau), h(\tau), H(\tau), \tilde{H}(\tau), Y(\tau)\}_{\tau=0}^{\infty}\) and prices \(\{\omega\}_{\tau=0}^{\infty}\) such that

i) conditions i)-iv) in Definition 18 are met, and

ii) all aggregate variables in per capita terms grow at a constant rate \(\gamma\).
In order to obtain the steady state growth in this economy, we first calculate the rate of growth of aggregate human capital: the sum of population growth rate plus the growth rate of average human capital. To obtain the latter we substitute (15) and (47) into (46),

\[ \tilde{H}(\tau) \kappa \zeta e^{\alpha r} = \int_{\tau-R}^{\tau-T} \mu \tilde{H}(t) T \zeta e^{\alpha t} m(t - \tau) \, dt. \]

If we take into account that along the balanced growth path \( \tilde{H}(t) = \tilde{H}(\tau)e^{-\gamma(\tau-t)} \) must hold, and we make the following change of variable \( z = \tau - t \), on recalling the survival probability (1) we can rewrite the above expression as,

\[ \frac{\mu T}{\kappa} \int_{T}^{R} e^{-(\gamma+n)z} \left[ \frac{\alpha - e^{-\beta z}}{\alpha - 1} \right] \, dz = 1, \] (50)

which implicitly characterizes the per capita growth rate \( \gamma \) as a function of \( \mu, T, \kappa, R, n, \alpha \) and \( \beta \). Ceteris paribus, from (50) one can see that for higher values of retirement age \( R \) or productivity in human capital production \( \mu \), the economy’s long run growth rate \( \gamma \) will be higher. If the education time length \( T \) becomes longer, then an ambiguous result shows up: i) aggregate human capital is enlarged (individual human capital stocks are higher), but ii) the share of active population becomes smaller. From (50) one has the following Proposition.

**Proposition 20** Assume that \( 0 < T < R \). If there is a per capita growth rate \( \gamma \) that satisfies (50), it must be unique.

**Proof:** See Appendix.

An open question is the convergence of this economy to the steady state. Equation (46) is exactly the same as the one obtained by Boucekkine et al. (2002). [See equation (23) in Boucekkine et al. (2002), p. 353.] Therefore, we refer the reader to that reference.\(^{17}\)

\(^{17}\)In essence Boucekkine et al. (2002) show that the dynamics of aggregate human capital is
Given that along balanced growth paths $R(\tau) = R$, $T(\tau) = T$, $\omega(\tau) = \omega$, and $\bar{H}(t) = \bar{H}(\tau)e^{-\gamma(\tau-t)}$, $\pi(t) = \pi(\tau)e^{-\pi(t)}$, taking into account (15), (17) and (18), and after a change of variable $z = \tau - t$, the equation for social security budget balance (49) can be rewritten as

$$s \int_T^R e^{-s+n}m(z)dz = \theta(1-s) \int R^J e^{-s+n}m(z)dz.$$  \hspace{1cm} (51)

Finally, if the solution for $T$ and $R$ happens to be interior, the payroll tax rate $s$ and the replacement rate $\theta$ which balance the social security budget are unique.

**Proposition 21** If $0 < T < R < J$, then there is a unique pair of payroll tax-replacement rates $(s, \theta)$ which satisfies the social security budget balance.

**Proof**: See Appendix. □

Thus, summing up, along the balanced growth equilibrium path conditions (34), (35), (50) and (51) must be met. This makes four non-linear equations in 4 unknowns: $T$, $R$, $\gamma$ and $\theta$ (for a given $s$). Unlike Boucekkine et al. (2002), it is not possible to obtain a relationship between $T$ and $\gamma$ in our model: $\gamma$ influences $T$ through social security budget balance which in turn affects $\theta$; and $T$ affects $\gamma$ through the growth rate equation. Therefore, it is not possible obtain a replica of their Proposition 3.4 [See Boucekkine et al. (2002), Proposition 3.4, p. 355.] Moreover, once the individual problem and the uniqueness of the steady state growth rate and the social security budget balance are solved analytically, all the results that follow in the next Section are strictly numerical.
4 A numerical example

In this Section we give values to the basic parameters to calculate numerically the steady state equilibrium for a benchmark case, and then we illustrate how our theoretical model responds to exogenous changes in life expectancy and social security policies. The summary of parameter values and the endogenous steady state values for the benchmark model are shown in Table 3. Two caveats are in order before continuing further.

First of all, the parameter set must be such that \( \eta_0 < \eta < \eta^* \). If \( \eta \leq \eta_0 \), condition (50) is not met; and if \( \eta \geq \eta^* \), from (51) one would have that \( \theta \) would be infinite. Therefore, we focus on interior solutions for \( T \) and \( R \) (i.e., \( 0 < T < R < J \)).

Second, our aim when choosing values is simply to illustrate the working and the main features of our model. In order to go further and obtain “realistic” conclusions from this exercise, more careful calibration work needs to be done. As a consequence of the simplicity of the demographics in our model (characterized by only 3 parameters: \( \alpha \), \( \beta \) and \( n \), or alternatively \( \kappa \)), it is difficult to reproduce observed moments of the age distribution. The model should include, for instance, physical capital, training during the active period, expenditure on education and, of course, more flexible functional forms (for utility and production). This alternative is not problem free, however: nonlinearities make it difficult (if not impossible) to obtain analytical results for the individual problem. And, more seriously, the uniqueness of the solution of the system of equations is at risk. [See note 12.]

Once we have set up the benchmark case, the experiment consists in generating

\[\text{[INSERT TABLE 3 AROUND HERE]}\]

\[\text{[18See Table 120, p. 86, in Vital Statistics, 2000 Statistical Abstract of the United States, United States Department of Commerce, Bureau of the Census. A remarkable point is that mortality rates, far from being constant, differ by sex and, mainly, by age. Once more, the need to choose between realism and analytical tractability shows up.}\]
different values for the life expectancy (between 40-50 and 100) with two alternatives: either through changes in $\alpha$, or in $\beta$. And with two possibilities in each case: constant $n$ and variable $\kappa$, or the other way around. This way we study the next Cases 1-4.\textsuperscript{19} The results are illustrated in Figures 5.1 – 5.4. We run an additional experiment: keeping the rest of the parameters constant, we analyze the response of our theoretical economy upon changes in the social security tax rate $s$.

4.1 Results

- **Case 1.** Figure 5.1 represents the responses of replacement rate, dependency ratio, education period, retirement age, life expectancy, per capita growth rate and average age of active workers upon changes in life expectancy caused by increments in $\alpha$, on the assumption that $n$ is kept constant and $\kappa$ variable.\textsuperscript{20}

The first result shows the existence of positive relationships between life expectancy $EV(0)$ and education $T$, retirement age $R$ and span of active period $R - T$: as seems reasonable, an extended lifetime allows individuals to augment the span of all phases. The longer the life expectancy, the higher the expected consumption expenditure and the need of income; additionally, as the mortality rate falls, the expected flow of future wages goes up: therefore, one should expect a longer education period. Considering the reaction of retirement age, and using Kalemli-Ozcan (2002a)’s terminology, the “horizon effect” dominates the “uncertainty effect”.\textsuperscript{21} Pension benefits are below net wages ($\theta < 1$), so that the above mentioned need for incomes also helps postpone retirement age. Moreover, the increase in life expectancy is greater than the increase in retirement age which, in turn, is greater

\textsuperscript{19}See equation (9).
\textsuperscript{20}The range of values for $\alpha$ goes from 3.0 to 11.9; $n$ and $\beta$ take on the same values as in the benchmark case, 0.010 and –0.0170, respectively. As a result, $EV$ ranges between 38.1 and 100.22, and $\kappa$ between 30.54 and 60.51.
\textsuperscript{21}A lower uncertainty about the possibility of really enjoying the retirement phase would favor plans for retiring earlier (rather than remaining active until the last moment).
than the increase in the education period. Therefore, proportions of individuals (at school, working and retired) change.

Along these lines, the positive response that we obtain in the dependency ratio $RD$ (the ratio of retirees to active workers) is the one we should expect. However, when we keep the social security contribution rate $s$ constant, the replacement rate $\theta$ slightly increases. Note the vintage nature of this model: if the economy’s growth rate is positive, not only does the age distribution across individuals matter (and the split between active and passive agents), but so does the growth rate of individual human capital $\gamma$. It can be shown that if in equation (51) $\gamma$ is set to 0, then the social security budget balance will be given by $s = \theta(1 - s)RD$. In this case, for a given $s$, a higher dependency ratio $RD$ implies, of course, a lower replacement rate $\theta$.

The inverted $U$ pattern for the growth rate-life expectancy relationship that we obtain is the same as the one De la Croix and Licandro (1999), Fuster (1999), Zhang, Zhang and Lee (2001)-(2003), Boucekkine et al. (2002), Zhang and Zhang (2003) or Echevarría (2004) find. Thus, for a low enough life expectancy, increases in life expectancy induce higher per capita growth rates; for high enough levels, however, an extended temporal horizon gives rise to lower per capita growth.

This is the consequence of several effects. First, the higher the $T$, the higher the individual human capital. But then individuals enter the labor market later (the proportion of working agents becomes smaller), and the labor force comprises older individuals who completed their education longer ago. Second, the higher the $R$, the higher the share of working individuals, but workers are also older. Actually, for the range of values of $\alpha$ which we consider, the average age of active workers $L$

\[22\text{Diamond and Gruber(1997), p. 1, estimate a dependency ratio (of individuals 65 or more to individuals between 20 and 40) of 0.14 in 1950, 0.21 in 1997, and predicts 0.36 in 2030 and 0.41 in 2070 for the US economy. For the same economy Galasso (1999) p. 712 estimates a ratio of 0.263 in 1990; and Gramlich (1999) estimates a ratio of 0.29 en 1999, and forecasts 0.56 in 2075.}
increases as life expectancy goes up. Third, individuals die later on average, so that the rate of depreciation of human capital falls.

- **Case 2.** Figure 5.2 represents a similar exercise but for changes in life expectancy caused by increments in $\beta$, holding $n$ constant. Comparing these results to those represented in Figure 5.1 (obtained under the assumption of higher $\alpha$’s) we can see that differences are minor, purely quantitative.\textsuperscript{23}

This result differs from Mateos (2003), who finds that a necessary condition for exogenous falls in mortality to induce higher levels of steady state income is that mortality drops are large enough among the youngest population [in our case, larger $\alpha$’s]. Zhang, Zhang and Lee (2001), however, find that drops in mortality associated with the oldest individuals [in our case, bigger $\beta$’s] may increase per capita income growth rate.

- **Case 3.** The exercise corresponding to changes in life expectancy via increments in $\alpha$, assuming this time that fertility ratio $1/\kappa$ rather than population growth is constant, is shown in Figure 5.3.\textsuperscript{24} The effects upon education $T$, retirement age $R$ and replacement rate $\theta$ are, from a qualitative point of view, similar to those obtained in Cases 1 and 2. However, the dependency ratio $RD$ slightly diminishes, and only a negative sloped relation between per capita growth rate $\gamma$ and life expectancy $EV(0)$ shows up.

- **Case 4.** Finally, Figure 5.4 represents the case of changes in life expectancy driven by increments in $\beta$, keeping $\kappa$ constant and allowing variable $n$.\textsuperscript{25} Differences

\textsuperscript{23}The range of values for $\beta$ goes from $-0.032$ to $-0.013$; $n$ and $\alpha$ take on the values of the benchmark case, 0.010 and 7.706, respectively. As a result, $EV$ ranges between 42.08 and 103.58, and $\kappa$ between 33.39 and 61.10.

\textsuperscript{24}The range of values for $\alpha$ goes from 4.25 to 11.9; $\kappa$ and $\beta$ are the same as in the benchmark case, 52.259 and $-0.0170$, respectively. Consequently, $EV$ ranges between 52.48 and 100.22, and $n$ between 0.0001 and 0.013. For values of $\alpha$ below 4.25, the algorithm used to solve (9) is unable to converge to reasonable values for $n$.

\textsuperscript{25}$\beta$ ranges between $-0.025$ and $-0.013$; $\kappa$ and $\alpha$ are the same as in the benchmark case, 52.259 and 7.706. $EV$ ranges between 53.86 and 103.58, and $n$ between 0.0009 and 0.014. For $\beta$ below $-0.025$, as happened in Case 3, the algorithm used to solve (9) for $n$ does not converge to reasonable
with respect to Case 3 are mainly quantitative.

[INSERT FIGURES 5.1-5.4 AROUND HERE.]

- **Social Security.**

  Figure 6 shows the effects of changes in the social security contribution rate $s$.\(^{26}\)

  The first result is a net discouraging effect upon human capital accumulation and retirement.

  A higher social security tax rate $s$ reduces the net wage $w(t) = (1 - s)\omega \mu \bar{H}(t)T$ [for a given $\bar{H}(t)$], so that incentives to devote a fraction of lifetime to education are reduced. Moreover, given that the replacement rate $\theta$ remains relatively unchanged, pension benefits (delayed wages) fall as well, so that incentives to lengthen the education phase are reduced additionally. Thus, it is not surprising that $T$ falls substantially.

  $R$ drops significantly when $s$ rises. As noted in subsection 3.3.1, $R = \eta(1 - \theta)(1 - s)\omega T$. Therefore, given that $\theta$ hardly changes, the discouraging effect on $T$ generates an even larger reduction in $R$ because $1 - s$ falls. Notice that for low enough (close to 0) $s$, $R$ is higher than $EV(0)$ (but less than $J$ at any rate, of course!): individuals would plan to retire only once they had survived life expectancy at birth. Given the reaction of $R$, it is easy to understand the response of $RD$: the higher the social security contribution rate, the higher the dependency ratio. This result is in line with Diamond and Gruber (1997), Coile and Gruber (2000), Fabel (1994) and Kalemli-Ozcan (2002a), among others.

  Given the behavior of $T$, $R$ and $RD$, the performance of $\gamma$ is as expected: the higher the social security contribution rate, the lower the rate of per capita growth.\(^ {27}\)

\(^{26}\) $s$ ranges between $s = 0.1\%$ and $s = 49.0\%$.

\(^{27}\) Nevertheless, the social security tax rate $s$ for which $\gamma$ attains the maximum is not zero, but slightly positive: in our numerical example, we obtain that $\gamma = 2.4\%$ for $s = 1.7\%$. Sánchez-Losada
Finally, focusing on social security, the replacement rate $\theta$ remains hardly unchanged. How is it possible that (upon increasing $s$) $RD$ goes up and $\theta$ stays almost constant? Remember, first, that $\theta$ represents the replacement rate defined on net wages $(1-s)w$; and, second, as noted in the beginning of this subsection, the social security budget balance condition depends on the dependency ratio, the contribution and the replacement rates and also the economy’s per capita growth rate $\gamma$.

As a result, one would expect the generosity of the pension scheme to fall. One way to measure this consists in calculating the ratio of the sum of present values of expected pension benefits to the sum of present values of expected social security contributions. Thus, from (1), (26), (15), (16), (17) and (18), and recalling that $\omega(\tau) = \omega$, one has

$$G = \frac{\int_{t+R}^{t+T} D(t, \tau)s \omega h(t)d\tau}{\int_{t+J}^{t+T} D(t, \tau)b(t)d\tau} = \frac{\theta(1-s) [\alpha(J - R) + e^{-\beta J} - e^{-\beta R}]}{s [\alpha(R - T) + e^{-\beta R} - e^{-\beta T}]}.$$ (52)

The relationship between $G$ and $s$ that we obtain is strictly decreasing. In fact, for low enough $s$ (in our numerical example $s < 0.31$), the $G$ that we obtain is higher than 1 (actuarially more than fair pension benefits).

**5 Conclusions**

In this work we studied the effects of changes in the mortality rate upon life expectancy, educational time investment, retirement age, human capital accumulation and economic growth in the presence of social security. Our starting point was Boucekkine et al. (2002), a continuous time growth, overlapping generations model in which individuals choose the optimal time length of education and retirement. (2000) obtains that if the capital income share is low enough, introducing unfunded social security may increase per capita growth rate. In both economies the existence of externalities in the human capital accumulation explains why positive social security may promote growth.
age. In this setup we have introduced a pay-as-you-go social security system in which pensions depend on past contributions made during the active life of workers.

The results obtained in the first part of the article are analytical.

i) We characterized the individual’s parameter space which establishes the type of solution for education length and retirement age (interior or corner).

ii) We proved the existence of, at most, one steady state per capita growth rate for interior solutions for education and retirement.

iii) We proved the existence of one unique steady state budget balance for the social security under the assumption of interior solutions for education and retirement.

The results obtained in the second part of the article are numerical. We ran some comparative statics exercises between steady states after changes in exogenous parameters: mortality rate falls as the origin of increments in life expectancy; and increments in the social security tax rate.

iv) The effects of increments in life expectancy upon education time length, retirement age and replacement rate are qualitatively the same whether they are caused by drops in the mortality rate of younger individuals or by drops in the mortality rate of the elderly.

v) The effects on the dependency ratio and the per capita growth rate, however, depend on the behavior of the rates of population growth and fertility:

v.1) if the population growth rate is kept constant and the fertility rate falls, the dependency ratio increases. Additionally, the per capita growth rate increases for low life expectancies, but diminishes for high ones.

v.2) if the population growth rate goes up and the fertility rate is kept constant, the dependency ratio and the per capita growth rate fall.

v.3) at any rate, the retirement age, the education time length and the social
security replacement rate always go up.

\textit{vi)} The higher the social security contribution rate, the greater the discouraging effects upon education time and labor supply. As a result, the retirement age falls and the dependency ratio rises. Additionally, the per capita growth rate falls.

\textit{vii)} If the social security contribution rate is raised, the replacement rate upon (net) wages remains almost constant.

\textit{viii)} Finally, higher social security contribution rates lower the generosity of the public pension system: the ratio of (expected and discounted) pension benefits to contributions to social security falls.
6 Appendix

Proof of Proposition 1. Substituting $x = 0$ into (42) one has that $M(0, \eta) = \frac{\theta}{\beta} [1 - \alpha(1 - \ln \alpha)]$. Let us define $f(\alpha) \equiv 1 - \alpha(1 - \ln \alpha)$. If $\theta > 0$ and $\beta < 0$, $M(0, \eta) < 0 \Leftrightarrow f(\alpha) > 0$. Note that $f(1) = 0$ and $f'(\alpha) = \ln \alpha > 0 \forall \alpha > 1$. Therefore, if $\alpha > 1$, then $f(\alpha) > 0$ and $M(0, \eta) < 0$. 

Proof of Lemma 2. Assume that $\beta < 0$, $\alpha > 1$ and $\eta_0 > 0$. From (43) one obtains that $K'(\eta) = \frac{\eta_0 \ln \alpha}{\eta^2 \beta} \left[2\alpha - \alpha \frac{\eta_0}{\eta} \left(2 + \frac{\eta_0}{\eta} \ln \alpha\right)\right] = \frac{\eta_0 \ln \alpha}{\eta^2 \beta} g(\alpha, \eta)$, where $g(\alpha, \eta) \equiv 2\alpha - \alpha \frac{\eta_0}{\eta} \left(2 + \frac{\eta_0}{\eta} \ln \alpha\right)$. Given the assumptions on $\alpha$, $\beta$ and $\eta_0$, $K'(\eta) > 0 \Leftrightarrow g(\alpha, \eta) < 0$. Thus, it is necessary to prove that if $\eta < \eta_0$, then $g(\alpha, \eta) < 0$. Notice the following facts:

i) $g(1, \eta) = 0$;

ii) $\partial g(\alpha, \eta) / \partial \alpha = 2 - \frac{\eta_0}{\eta} \alpha \frac{\eta_0 - \eta}{\eta} \left(3 + \frac{\eta_0}{\eta} \ln \alpha\right) \equiv h(\alpha, \eta)$;

iii) $h(\alpha, \eta_0) = -1 - \ln \alpha$;

iv) $h(1, \eta_0) = -1$;

v) $\partial h(\alpha, \eta_0) / \partial \alpha = -1/\alpha < 0$; therefore,

vi) $g(\alpha, \eta_0) < 0$. We need to check how $g(\alpha, \eta)$ behaves for values of $\eta$ less than $\eta_0$. It is straightforward to check that:

vii) $\partial g(\alpha, \eta) / \partial \eta = \frac{\eta_0}{\eta^2} \alpha \frac{\eta_0 - \eta}{\eta} \left(3 + \frac{\eta_0}{\eta} \ln \alpha\right) \ln \alpha > 0$.

Therefore, from vii) and viii) one has that if $\eta < \eta_0$, then $g(\alpha, \eta) < 0 \Leftrightarrow K'(\eta) > 0$. 

Proof of Lemma 3. Assume that $\alpha > 1$, $\beta < 0$ and $\eta_0 > 0$.

i) Trivially, from (43) $K(\eta_0) = \frac{1}{\beta} [-\alpha \ln \alpha + \alpha(\ln \alpha - 1) + \alpha] = 0$.

ii) Finally, from the previous proof one has that

$K'(\eta) = \frac{\eta_0 \ln \alpha}{\eta^2 \beta} \left[2\alpha - \alpha \frac{\eta_0}{\eta} \left(2 + \frac{\eta_0}{\eta} \ln \alpha\right)\right]$. Evaluating $K'(\eta)$ in $\eta = \eta_0$, one has

$K'(\eta_0) = \frac{\eta_0 \ln \alpha}{\beta \eta_0} \left[2\alpha - \alpha (2 + \ln \alpha)\right] = -\frac{\alpha (\ln \alpha)^2}{\eta_0}$ > 0.

Proof of Lemma 4. From Lemma 2 and part i) in Lemma 3, it is trivial. 

Proof of Lemma 5. Assume that $\beta < 0$, $\alpha > 1$ and $\eta_0 > 0$.

i) After some tedious algebra, it can be shown from (43) that

$K(2\eta_0) = \frac{\alpha^{1/2}}{\beta} \left(\frac{\ln \alpha}{2} + 1 - \alpha^{1/2}\right) = \frac{\alpha^{1/2}}{\beta} i(\alpha)$, where $i(\alpha) \equiv \frac{\ln \alpha}{2} + 1 - \alpha^{1/2}$. Given the negative sign of $\beta$, then $K(2\eta_0) > 0 \Leftrightarrow i(\alpha) < 0$. Notice that, first, $i(1) = 0$.
and, second, \(i'(\alpha) = \frac{\alpha^{-1/2}}{2} (\alpha^{-1/2} - 1) < 0 \Leftrightarrow \alpha > 1\). Therefore, if \(\alpha > 1\), then \(i(\alpha) < 0\) and \(K(2\eta_0) > 0\).

ii) Finally, from the previous proof, one has that
\[
K'(\eta) = \frac{n_0 \ln \alpha}{\eta^3} \left[2\alpha - \alpha^{\frac{3\alpha}{\eta}} \left(2 + \frac{n_0}{\eta} \ln \alpha\right)\right].
\]
Evaluating \(K'(\eta)\) at \(\eta = 2\eta_0\), one has
\[
K'(2\eta_0) = \frac{\ln \alpha}{4\eta_0^3} i(\alpha),
\]
where \(j(\alpha) = 2\alpha - \alpha^{\frac{3\alpha}{\eta}} \left(2 + \frac{n_0}{\eta} \ln \alpha\right)\). Given the restrictions upon \(\beta\) and \(\eta_0\), then \(K'(2\eta_0) < 0 \Leftrightarrow j(\alpha) > 0\). Note the following facts:

i.1) \(j(1) = 0\);

ii.2) \(j'(\alpha) = 2 - \alpha^{-1/2} \left(\frac{3}{2} + \frac{\ln \alpha}{\alpha}\right)\);

ii.3) \(j'(1) = 1/2\); and,

ii.4) \(j''(\alpha) = \alpha^{-3/2} \left(\frac{\ln \alpha}{8} + \frac{1}{2}\right) > 0\) if \(\alpha > 1\).

Therefore, if \(\alpha > 1\), then \(j(\alpha) > 0\) and \(K'(2\eta_0) < 0\).

Proof of Lemma 6. Assume that \(\beta < 0\), \(\alpha > 1\) and \(\eta_0 > 0\). From (43) one obtains that \(K'(\eta) = \frac{n_0 \ln \alpha}{\beta \eta^2} g(\alpha, \eta)\), where \(g(\alpha, \eta) \equiv 2\alpha - \alpha^{\frac{3\alpha}{\eta}} \left(2 + \frac{n_0}{\eta} \ln \alpha\right)\), continuous in \(\eta\) for all \(\eta > 0\). Therefore, \(K'(\eta) = 0 \Leftrightarrow g(\alpha, \eta) = 0\). Moreover, \(g(\alpha, \eta) = 0 \Leftrightarrow l(\alpha, \eta) = m(\alpha, \eta)\), where \(l(\alpha, \eta) \equiv 2\alpha^{\frac{\eta - n_0}{n_0}}\) and \(m(\alpha, \eta) \equiv 2 + \frac{n_0}{\eta} \ln \alpha\).

It can be shown that:

i) \(l(\alpha, \eta)\) is strictly increasing in \(\eta\);

ii) \(l(\alpha, \eta_0) = 2\);

iii) \(l(\alpha, 2\eta_0) = 2\alpha^{1/2}\);

iv) \(m(\alpha, \eta)\) is strictly decreasing in \(\eta\);

v) \(m(\alpha, \eta_0) = 2 + \ln \alpha\);

vi) \(m(\alpha, 2\eta_0) = 2 + \frac{1}{2} \ln \alpha\);

vii) \(m(\alpha, \eta_0) > l(\alpha, \eta_0)\), whose proof is trivial;

viii) \(m(\alpha, 2\eta_0) < l(\alpha, 2\eta_0)\). Note that \(m(\alpha, 2\eta_0) < l(\alpha, 2\eta_0) \Leftrightarrow n(\alpha) \equiv 2 + \frac{1}{2} \ln \alpha - 2\alpha^{1/2} < 0\), and also that \(n(1) = 0\) and \(n'(\alpha) = \alpha^{-1/2} \left(\frac{1}{2\alpha^{1/2}} - 1\right) < 0\) if \(\alpha > 1\). Therefore, if \(\alpha > 1\), then \(n(\alpha) < 0\); equivalently, \(m(\alpha, 2\eta_0) < l(\alpha, 2\eta_0)\).

To sum up, from i), iv), vii) and viii) one obtains that there is one unique \(\hat{\eta} \in (\eta_0, 2\eta_0)\) such that \(l(\alpha, \hat{\eta}) = m(\alpha, \hat{\eta})\), that is to say, \(g(\alpha, \hat{\eta}) = 0\); equivalently, \(K'(\hat{\eta}) = 0\).

Proof of Lemma 7. Assume that \(\beta < 0\), \(\alpha > 1\) and \(\eta_0 > 0\). From (43) one has that \(K'(\eta) = \frac{n_0 \ln \alpha}{\beta \eta^2} g(\alpha, \eta)\), where \(g(\alpha, \eta) \equiv 2\alpha - \alpha^{\frac{3\alpha}{\eta}} \left(2 + \frac{n_0}{\eta} \ln \alpha\right)\) is continuous in \(\eta\) for all \(\eta > 0\). We have proved in Lemma 5 that \(K'(2\eta_0) < 0\), that is, \(g(\alpha, 2\eta_0) > 0\). In addition, \(\partial g(\alpha, \eta)/\partial \eta = \frac{n_0 \alpha^{-\frac{3\alpha}{\eta}}}{3 + \frac{n_0}{\eta} \ln \alpha} \ln \alpha > 0\). Therefore, if \(\eta > 2\eta_0\), then \(g(\alpha, \eta) > 0\) and \(K'(\eta) < 0\).
Proof of Lemma 8. Assume that $\beta < 0$ and $\alpha > 1$. From (43) one has that $\lim_{\eta \to \infty} K(\eta) = \frac{a}{\beta} \left( \ln \alpha + \frac{1}{\alpha} - 1 \right) \equiv \frac{a}{\beta} o(\alpha)$, where $o(\alpha) \equiv \ln \alpha + \frac{1}{\alpha} - 1$. Therefore, $\lim_{\eta \to \infty} K(\eta) < 0 \iff o(\alpha) > 0$. Note that $o(1) = 0$ and $o'(\alpha) = \frac{1}{\alpha} (1 - \frac{1}{\alpha}) > 0$. Therefore, $\lim_{\eta \to \infty} K(\eta) < 0$. $\blacksquare$

Proof of Corollary 9. From Lemma 5, Lemma 7 and Lemma 8 the proof is trivial. $\blacksquare$

Proof of Corollary 10. From Lemmas 4-6 and Corollary 9 the proof is trivial. $\blacksquare$

Proof of Proposition 11. From Proposition 1 and Corollary 10 the result is trivial. $\blacksquare$

Proof of Proposition 12. Assume that $\beta < 0$, $\alpha > 1$ and $\eta_0 > 0$. From Proposition 11 one has that there is at least one $x \in (0, T_{\max}(\eta))$ such that $M(x, \eta) = 0$. To prove that it is unique, we split $M(x, \eta)$ as the difference of two functions $p(x, \eta)$ and $q(x, \eta)$, and prove that one unique intersection point exists between them.

i) $M(x, \eta)$ can be rewritten as $M(x, \eta) = p(x, \eta) - q(x, \eta)$, where $p(x, \eta) \equiv 2\alpha x - \frac{1}{\beta} e^{-\beta \frac{a}{\eta_0} x} - \frac{a}{\beta} e^{-\beta \eta_0 x}$, and $q(x, \eta) \equiv x e^{-\beta x} + \theta \alpha J + (1 - \theta) \alpha x \frac{a}{\eta_0} - \frac{1}{\beta} e^{-\beta x}$; 

ii) using the definition of $J$, $p(0, \eta) = -\frac{1}{\beta} - \frac{a}{\beta} (\alpha - 1) > 0$;

iii) given the definitions of $J$ and $T_{\max}(\eta)$, $p[T_{\max}(\eta), \eta] = -\frac{a}{\beta} \left( \frac{2 \eta_0}{\eta} \ln \alpha + 1 \right) > 0$;

iv) $\partial p(x, \eta)/\partial x = 2 \alpha + \frac{1 - \theta \eta}{\eta_0} e^{-\beta \frac{a}{\eta_0} x} > 0$, that is, $p(x, \eta)$ is strictly increasing in $x$;

v) $\partial^2 p(x, \eta)/\partial x^2 = \frac{(1 - \theta) \eta_0^2 \beta}{\eta_0} e^{-\beta \frac{a}{\eta_0} x} > 0$, that is, $p(x, \eta)$ is strictly convex in $x$;

vi) $q(0, \eta) \equiv -\frac{1}{\beta} (\theta \alpha \ln \alpha + 1) > 0$ and $\alpha > 1$;

vii) given the definitions of $J$ and $T_{\max}(\eta)$,

$\eta$ $T_{\max}(\eta), \eta] = -\frac{1}{\beta} \left[ \left( \alpha + \frac{a}{\eta_0} \frac{a}{\eta_0} \frac{a}{\eta_0} \frac{a}{\eta_0} \right) \ln \alpha + \alpha \frac{a}{\eta_0} \right] > 0$;

viii) $\partial q(x, \eta)/\partial x = e^{-\beta x} (2 - x \beta) + (1 - \theta) \alpha \frac{a}{\eta_0} > 0$, that is, $q(x, \eta)$ is strictly increasing in $x$;

ix) $\partial^2 q(x, \eta)/\partial x^2 = e^{-\beta x} (x \beta^2 - 3 \beta) > 0$, that is, $q(x, \eta)$ is strictly convex in $x$;

x) $p(0, \eta) < q(0, \eta) \iff \alpha \ln \alpha - \alpha + 1 > 0$; see expression $o(\alpha)$ in Lemma 8;

xi) $p[T_{\max}(\eta), \eta] > q[T_{\max}(\eta), \eta] \iff M[T_{\max}(\eta), \eta] \equiv K(\eta) > 0$ if $\eta \in (\eta_0, \eta^*)$. [See Corollary 10.]

Therefore, $p(x, \eta)$ and $q(x, \eta)$ cross each other only once between $x = 0$ and $x = T_{\max}(\eta)$, that is, there exists one unique $x \in (0, T_{\max}(\eta))$ such that $p(x, \eta) = q(x, \eta)$.
\( \Leftrightarrow M(x, \eta) = 0 \). The following plot in Figure 7 can help us understand the proof.

[INSERT FIGURE 7 AROUND HERE]  ■

**Proof of Proposition 13.** Given (1), \( x = T \) is a solution to (35) if and only if \( \tilde{M}(x, \eta) = 0 \), where

\[
\tilde{M}(x, \eta) \equiv x \left( \alpha - e^{-\beta x} \right) + \alpha \left[ x - \min \left\{ \frac{\eta x}{\eta_0}, J \right\} \right] + e^{-\beta x} - e^{-\beta \min \left\{ \frac{\eta x}{\eta_0}, J \right\}} + \theta \alpha \left[ \min \left\{ \frac{\eta x}{\eta_0}, J \right\} - J \right] + \theta \frac{e^{-\beta \min \left\{ \frac{\eta x}{\eta_0}, J \right\}} - e^{-\beta J}}{\beta}.
\]

The strategy of the proof will follow these steps:

i) first, we will prove that \( \tilde{M}(J, \eta) = 0 \);

ii) second, we will prove that for values of \( x \) less than and close enough to \( J \) we will have that \( \tilde{M}(x, \eta) > 0 \);

iii) third, we will prove that \( \tilde{M}(0, \eta) < 0 \);

iv) fourth, given that \( \tilde{M}(x, \eta) \) is continuous, at least one \( x \in (0, J) \) exists such that \( \tilde{M}(x, \eta) = 0 \); and

v) fifth, if such an \( x \) is unique and interior, \( 0 < x < T_{\text{max}}(\eta) < J \), then \( K(\eta) > 0 \), which implies that \( \eta \in (\eta_0, \eta_*) \) as we have proven in Corollary 10.

In subsection 3.3.1 we have proven that if the solution is interior, then it must be the case that \( \eta > \eta_0 \).

i) If so, \( \min \left\{ \eta J, J \right\} = J \). Therefore, from (53) and (2) one has that \( \tilde{M}(J, \eta) = J(\alpha - e^{-\beta J}) = 0 \).

ii) Similarly, for \( x \) less than but close enough to \( J \), \( \min \left\{ \frac{\eta x}{\eta_0}, J \right\} = J \). Therefore, from (53) one has that

\[
\tilde{M}(x, \eta) \equiv x \left( \alpha - e^{-\beta x} \right) + \alpha(x - J) + \frac{e^{-\beta x} - e^{-\beta J}}{\beta} \Rightarrow
\]

\[
\Rightarrow \frac{\partial \tilde{M}(x, \eta)}{\partial x} = 2(\alpha - e^{-\beta x}) + \beta xe^{-\beta x} \Rightarrow
\]

\[
\Rightarrow \lim_{x \to J-} \frac{\partial \tilde{M}(x, \eta)}{\partial x} = 2(\alpha - e^{-\beta J}) + \beta xe^{-\beta J} = -\alpha \ln \alpha < 0,
\]

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given that $\alpha > 1$. Thus, given $i)$, for $x$ less than but close enough to $J$ one has that \( \hat{M}(x, \eta) > 0 \).

iii) From (53) one obtains that
\[
\hat{M}(0, \eta) = \frac{\theta}{\beta} [1 + \alpha (\ln \alpha - 1)] < 0.
\]
The previous inequality is immediately checked if we assume that $\beta < 0$ and define $f(\alpha) \equiv 1 + \alpha (\ln \alpha - 1)$, so that $f(1) = 0$, $f'(\alpha) = \ln \alpha > 0$ and, therefore, $f(\alpha) > 0$ for $\alpha > 1$.

iv) Thus, given that $\hat{M}(x, \eta)$ is continuous, from ii) and iii) one has that at least one $x^* \in (0, J)$ exists such that $\hat{M}(x, \eta) = 0$;

v) Given (41), from (42), (43) and (53) it can be obtained
\[
\hat{M}(T_{\max}(\eta), \eta) = M[T_{\max}(\eta), \eta] = K(\eta).
\]

If the answer of item iv) is unique and interior, that is to say, $0 < x^* < T_{\max}(\eta) < J$, then $\hat{M}[T_{\max}(\eta), \eta] > 0$. In this case, given the equality in the previous expression, $K(\eta) > 0$ and for this to happen we have seen that it is necessary that $\eta \in (\eta_0, \eta^*)$.

[See Corollary 10.]

**Proof of Proposition 14.** The strategy of the proof is as follows:

i) first, we prove that $T = T_{\max}(\eta^*) \equiv \frac{J\eta_0}{\eta^*}$ and $R = J$ is a solution to (37) and (39);

ii) second, we prove that it is the unique one which meets the condition $0 < T < J$.

i) Assume that $\beta < 0$, $\alpha > 1$, $\eta = \eta_0 > \eta > 0$. From Corollary 10 one has that $K(\eta_0) = 0$. Given the definition of $K(\eta) \equiv M[T_{\max}(\eta), \eta]$, one has that $M[T_{\max}(\eta_0), \eta] = 0$; equivalently, $T_{\max}(\eta_0) \equiv \frac{J\eta_0}{\eta^*} < J$ is a solution to (39). In this case, from (37) one has that $R = J$. In sum, $T = T_{\max}(\eta_0) \equiv \frac{J\eta_0}{\eta^*}$ and $R = J$ satisfy (37) and (39), so that corner and interior solutions coincide.

ii) To see that it is the unique one such that $0 < T < J$ we have to prove that $Tm(T) = \int_T^R m(\tau)d\tau + \theta \int_R^J m(\tau)d\tau = \int_T^J m(\tau)d\tau$ has one unique solution $T < J$ [because, trivially, $Jm(J) = \int_J^J m(\tau)d\tau = 0$ by definition of $J$ in (2)]. Given (1) and (2), it can be shown after some algebra that condition $Tm(T) = \int_T^J m(\tau)d\tau$ is equivalent to $r(T, \alpha, \beta) = 0$, where $r(T, \alpha, \beta) \equiv \left( T - \frac{1}{\beta} \right) \left( \alpha - e^{-\beta T} \right) - \alpha (J - T)$. Note that

ii.1) $r(T, \alpha, \beta)$ is continuous in $T$ and in $\alpha$;

ii.2) by definition of $J$, $r(0, \alpha, \beta) = \frac{-1}{\beta} (\alpha - 1 - \alpha \ln \alpha)$;
\[ \begin{align*}
  \text{ii.3) } r(0, 1, \beta) & = 0; \\
  \text{ii.4) } \partial r(0, \alpha, \beta)/\partial \alpha & = \frac{\ln \alpha}{\beta} < 0 \text{ if } \alpha > 1 \text{ and } \beta < 0 \text{ and that, therefore,} \\
  \text{ii.5) } r(0, \alpha, \beta) & < 0 \text{ if } \beta < 0 \text{ and } \alpha > 1.
\end{align*} \]

\[ \begin{align*}
  \text{ii.6) Given the definition of } J, \quad r(J, \alpha, \beta) & = 0. \\
  \text{ii.7) } \partial r(T, \alpha, \beta)/\partial T & = 2\alpha - (2 - \beta T)e^{-\beta T} \text{ and, therefore,} \\
  \text{ii.8) if } T = 0, \text{ then } \partial r(T, \alpha, \beta)/\partial T & = 2(\alpha - 1) > 0 \text{ if } \alpha > 1, \text{ and} \\
  \text{ii.9) if } T = J, \text{ then } \partial r(T, \alpha, \beta)/\partial T & = -\alpha \ln \alpha < 0 \text{ if } \alpha > 1.
\end{align*} \]

\[ \begin{align*}
  \text{ii.10) To sum up, from ii.1, ii.5, ii.6, ii.8, and ii.9, one has that at least one } d \in (0, J) \text{ exists such that } \partial r(T, \alpha, \beta)/\partial T = 0 \text{ for } T = d. \\
  \text{ii.11) Finally, it can be shown that } \partial^2 r(T, \alpha, \beta)/\partial T^2 = e^{-\beta T}\beta(3 - \beta T) < 0 \text{ if } \beta < 0, \text{ that is, } r(T, \alpha, \beta) \text{ is strictly concave in } T.
\end{align*} \]

\[ \begin{align*}
  \text{ii.12) Therefore, from ii.10 and ii.11, one has that there is one unique } d \in (0, J) \\
  \text{for which } \partial r(T, \alpha, \beta)/\partial T = 0 \text{ for } T = d \text{ and, therefore,} \\
  \text{ii.13) from ii.1, ii.5, ii.6 and ii.12, one has that one unique } T \text{ [where } 0 < T < d < J \text{] exists such that } r(T, \alpha, \beta) = 0 \iff T m(T) = \int_T^J m(\tau)d\tau.
\end{align*} \]

Figure 8 illustrates this proof.

\[ \begin{align*}
  \text{Proof of Proposition 15.} \\
  \text{This is the strategy of the proof:} \\
  \text{i) first, we will show that there is no } x \in [0, T_{\max}(\eta)] \text{ such that (35) has solution or that, equivalently, } \hat{M}(x, \eta) \text{ defined in (53) takes the value zero.} \\
  \text{ii) second, we will show that } \hat{M}(J, \eta) = 0, \text{ therefore } J \text{ is a solution.} \\
  \text{iii) third, we will prove that one unique } x \in (T_{\max}(\eta), J) \text{ exists such that } \hat{M}(x, \eta) = 0, \text{ where } x = T_{\max}(\eta_*) \equiv \frac{J_{\eta_*}}{\eta_*}, \text{ so that } T = T_{\max}(\eta_*) \equiv \frac{J_{\eta_*}}{\eta_*}, \text{ and } R = J.
\end{align*} \]

\[ \begin{align*}
  \text{i.1) Assume that } \beta < 0, \alpha > 1 \text{ and } \eta > \eta_* > \eta_0 > 0. \text{ Given (41), from (42), (43) and (53) it can be checked that} \\
  \hat{M}[T_{\max}(\eta), \eta] = M[T_{\max}(\eta), \eta] \equiv K(\eta).
\end{align*} \]

According to Corollary 10, if } \eta > \eta_*, \text{ then } K(\eta) < 0. \text{ Therefore, } \hat{M}[T_{\max}(\eta), \eta] < 0.

\[ \begin{align*}
  \text{i.2) } \hat{M}(x, \eta) \text{ defined in (53) can be rewritten as } \hat{M}(x, \eta) = \hat{p}(x, \eta) - \hat{q}(x, \eta), \text{ where} \\
  \hat{p}(x, \eta) \equiv 2\alpha x - \frac{(1 - \theta)}{\beta} e^{-\beta \min\{\frac{\eta_*}{\eta_0}, J\}} - \frac{\theta}{\beta} e^{-\beta J}, \text{ and} \\
\end{align*} \]
\[
\dot{q}(x, \eta) \equiv xe^{-\beta x} + \alpha \theta J + \alpha(1 - \theta) \min \left\{ \frac{\eta x}{\eta_0}, J \right\} - \frac{e^{-\beta x}}{\beta}.
\]

i.3) Let us assume values of \( x \in [0, T_{\text{max}}(\eta)] \). In this case, \( \min \left\{ \frac{\eta x}{\eta_0}, J \right\} = \frac{\eta x}{\eta_0} \), so that

\[
\dot{p}(x, \eta) \equiv 2\alpha x - \frac{(1 - \theta)}{\beta} e^{-\beta \frac{\eta x}{\eta_0}} - \frac{\theta}{\beta} e^{-\beta J}, \quad \text{and}
\]

\[
\dot{q}(x, \eta) \equiv xe^{-\beta x} + \alpha \theta J + \alpha(1 - \theta) \frac{\eta x}{\eta_0} - \frac{e^{-\beta x}}{\beta}.
\]

i.4) \( \dot{p}(0, \eta) = -\frac{1}{\beta} \left[ 1 + \theta(\alpha - 1) \right] > 0 \).

i.5) \( \dot{p} [T_{\text{max}}(\eta), \eta] = -\frac{\alpha}{\beta} \left[ 1 + 2\frac{\alpha}{\eta} \ln \alpha \right] > 0 \).

i.6) \( \frac{\partial \dot{p}(x, \eta)}{\partial x} = 2\alpha + \frac{(1 - \theta)\eta}{\eta_0} e^{-\beta \frac{\eta x}{\eta_0}} > 0; \ \dot{p}(x, \eta) \) is strictly increasing for \( x \in (0, T_{\text{max}}(\eta)) \).

i.7) \( \frac{\partial^2 \dot{p}(x, \eta)}{\partial x^2} = -\frac{(1 - \theta)\beta \eta^2}{\eta_0^2} e^{-\beta \frac{\eta x}{\eta_0}} > 0; \ \dot{p}(x, \eta) \) is strictly convex for \( x \in (0, T_{\text{max}}(\eta)) \).

i.8) \( \dot{q}(0, \eta) = -\frac{1}{\beta} \left[ 1 + \alpha \theta \ln \alpha \right] > 0 \).

i.9) \( \dot{q} [T_{\text{max}}(\eta), \eta] = \frac{1}{\beta} \left[ \alpha \frac{\eta_0}{\eta} + \left( \alpha + \frac{\eta_0}{\eta} \ln \alpha \right) \right] \ln \alpha > 0 \).

i.10) \( \frac{\partial \dot{q}(x, \eta)}{\partial x} = (2 - \beta x) e^{-\beta x} + \alpha(1 - \theta) \frac{\eta}{\eta_0} > 0; \ \dot{q}(x, \eta) \) is strictly increasing for \( x \in (0, T_{\text{max}}(\eta)) \).

i.11) \( \frac{\partial^2 \dot{q}(x, \eta)}{\partial x^2} = (x^2 - 3\beta) e^{-\beta x}; \ \dot{q}(x, \eta) \) is strictly convex for \( x \in (0, T_{\text{max}}(\eta)) \).

i.12) From i.1) and i.8) one has that \( \dot{p}(0, \eta) - \dot{q}(0, \eta) = -1 + \alpha(1 - \ln \alpha) < 0 \).

i.13) From i.1) and i.2) one has that \( \dot{M} [T_{\text{max}}(\eta), \eta] \equiv \dot{p} [T_{\text{max}}(\eta), \eta] - \dot{q} [T_{\text{max}}(\eta), \eta] \) \( < 0 \).

i.14) Therefore, from i.4)-i.13) one has that \( \dot{p}(x, \eta) \) and \( \dot{q}(x, \eta) \) do not cross each other at any \( x \in [0, T_{\text{max}}(\eta)] \); equivalently, no \( x \in [0, T_{\text{max}}(\eta)] \) exists such that \( \dot{M}(x, \eta) = 0 \). And if there is some \( x \) for which \( \dot{M}(x, \eta) = 0 \), then \( x \in (T_{\text{max}}(\eta), J) \).

ii) Given that \( \eta > \eta_0 > \eta_0 \), \( \min \left\{ \frac{\eta x}{\eta_0}, J \right\} = J \) and, therefore, from (53) one has that \( \dot{M}(J, \eta) = 0 \), that is \( x = J \) is a solution to (53).

iii) We are going to prove that one unique \( T \in (T_{\text{max}}(\eta), J) \) exists such that \( \dot{M}(x, \eta) = 0 \), where \( T = T_{\text{max}}(\eta_0) \equiv \frac{J_{\eta_0}}{\eta_0} \), this way \( T > T_{\text{max}}(\eta_0) \), and \( R = J \).

iii.1) From (34) one has that \( R = \min \left\{ \frac{\eta T}{\eta_0}, J \right\} \) and from (41) \( T_{\text{max}}(\eta) \equiv \frac{J_{\eta_0}}{\eta} \). By hypothesis, \( T > T_{\text{max}}(\eta) \). Therefore \( R = J \).

iii.2) From iii.1), (34) and (35) one has that \( T \) must satisfy

\[
T m(T) = \int_T^J m(\tau)d\tau.
\]

And this is, precisely, Case 1 studied in Proposition 14. Therefore, one unique \( T < J \) exists which satisfies the previous equation, so that \( T = T_{\text{max}}(\eta_0) \equiv \frac{J_{\eta_0}}{\eta_0} < J \) if
\( \eta_* > \eta_0 \). And, additionally, \( T > T_{\text{max}}(\eta) \) because by definition of \( T_{\text{max}}(\eta) \) in (41), \( \frac{J_{\eta_0}}{\eta_*} > \frac{J_{\eta}}{\eta} \Leftrightarrow \eta > \eta_* \). In short, if \( \eta > \eta_* \), then \( T = T_{\text{max}}(\eta_*) \) and \( R = J \). 

**Proof of Proposition 16.** If \( \eta = \eta_0 \), from (34) one has that \( R = T \leq J \), because \( T \) cannot be greater than \( J \). At any rate, \( R = T \), so that social security tax revenues are zero.

a) Assume that social security budget balance is required so that \( \theta = 0 \). From (35) we obtain that

\[
Tm(T) = 0. \tag{54}
\]

The strategy of the proof consists of two steps: first, we prove that (54) admits only two solutions; and, second, we prove that the indirect utility function attains a higher value in one of the two.

Equation (54) has 2 possible solutions: \( T = 0 \) and \( T = J \). To check that \( Tm(T) = 0 \) admits only these 2 solutions, notice that \( Tm(T) = 0 \Leftrightarrow u(T) \equiv T(\alpha - e^{-\beta T}) = 0. \)

i) \( u(0) = 0 \): 0 is a solution;

ii) \( u(J) = 0 \): \( J \) is a solution;

iii) \( u'(T) = e^{-\beta T}(\beta T - 1) + \alpha; \)

iv) \( u'(0) = \alpha - 1 > 0 \) (\( u \) is increasing in \( T = 0 \));

v) \( u'(J) = -\ln \alpha < 0 \) (\( u \) is decreasing in \( T = J \)); and

vi) \( u''(T) = \beta e^{-\beta T}(2 - \beta T) < 0 \) [i.e., \( u(T) \) is strictly concave and, therefore, no \( T \in (0, J) \) exists which is a solution to (54)].

To sum up, there are two solutions: \( 0 = T = R < J \), and \( 0 < T = R = J \). The situation is described in Figure 9.

[INSERT FIGURE 9 AROUND HERE]

Finally, we will prove that the indirect utility function is higher at \( T = 0 \) than at \( T = J \).

i) Substituting the rest of equality restrictions into the first restriction of (20); ii) taking into account that \( \omega(\tau) = \bar{\omega}(t) = \omega, T(t) = T, R(t) = \eta T/\eta_0 \), and that \( m(\tau - t) = D(t, \tau) \) [given (1)];

iii) solving for \( \int_t^{t+J} C(t, \tau)m(\tau - t)d\tau \) and substituting into the lifetime utility function (14);

iv) recalling that \( \eta \equiv \mu \phi \), we obtain the indirect utility function for a solution in which \( R = T\eta/\eta_0 \) (without loss of generality, we assume an individual born at \( t = 0 \), and
v) assuming \( \eta = \eta_0 \) and \( \theta = 0 \), we obtain
\[
V(T, \eta) = -\frac{\tilde{H}}{\phi} \int_0^T \tau \left( e^{-\beta \tau} - \frac{\alpha}{1 - \alpha} \right) d\tau,
\]
which, trivially, is decreasing in \( T \). Therefore, solution \( 0 = T = R < J \) is preferred to solution \( 0 < T = R = J \).

b) Assume now that a social security budget deficit is allowed and \( \theta > 0 \). From (35) one has
\[
T_m(T) = \theta \int_T^J m(\tau) d\tau. \tag{55}
\]
As before, the strategy of the proof requires two steps: first, to prove that (55) has only two solutions; and, second, that the indirect utility function is unambiguously greater in one of the two.

i) (55) has two solutions: \( T = J \), and \( 0 < T(\theta) < J \). As for the previous case of \( \theta = 0 \), \( T_m(T) = \theta \int_T^J m(\tau) d\tau \Leftrightarrow v(T, \alpha, \beta, \theta) = \left( T - \frac{\theta}{\beta} \right) (\alpha - e^{-\beta T}) - \theta \alpha (J - T) = 0 \), where \( v \) is continuous in \( T \). Therefore,

i.1) \( v(J, \alpha, \beta, \theta) = 0 \); (\( v \) equals zero in \( T = J \));

i.2) \( v(0, \alpha, \beta, \theta) = -\frac{\theta}{\beta} (\alpha - 1 - \alpha \ln \alpha) \);

i.3) from i.2) \( v(0, 1, \beta, \theta) = 0 \);

i.4) from i.2) \( \partial v(0, \alpha, \beta, \theta)/\partial \alpha = \frac{\theta \ln \alpha}{\beta} < 0 \) if \( \alpha > 1 \) and \( \beta < 0 \);

i.5) from i.3) and i.4) one has that \( v(0, \alpha, \beta, \theta) < 0 \) if \( \alpha > 1 \) and \( \beta < 0 \) (\( v \) is negative at \( T = 0 \));

i.6) from the definition of \( v \), one has that \( \partial v(T, \alpha, \beta, \theta)/\partial T = (1 + \theta) \alpha - e^{-\beta T} (1 - \beta T + \theta) \);

i.7) from i.6) one has \( \partial v(T, \alpha, \beta, \theta)/\partial T|_{T=0} = (1 + \theta) (\alpha - 1) > 0 \) if \( \alpha > 1 \) (\( v \) is increasing at \( T = 0 \));

i.8) from i.6) one has \( \partial v(T, \alpha, \beta, \theta)/\partial T|_{T=J} = -\alpha \ln \alpha < 0 \) if \( \alpha > 1 \) (\( v \) is decreasing in \( T = J \));

i.9) \( \partial^2 v(T, \alpha, \beta, \theta)/\partial T^2 = \beta e^{-\beta T} (2 + \theta - \beta T) < 0 \) (\( v \) is strictly concave);

i.10) finally, from i.1), i.5), i.7), i.8) and i.9) one has that one unique \( T(\theta) \in (0, J) \) exists which satisfies (55). Therefore, there are two solutions: \( 0 < T(\theta) = R < J \), and \( 0 < T = R = J \). The argument is described in Figure 10.

\[\text{[INSERT FIGURE 10 AROUND HERE]}\]

Notice that consumption is zero in both cases, but leisure is positive only in the first case. Therefore, solution \( 0 < T(\theta) = R < J \) is preferred to solution \( 0 < T = R = J \).
Proof of Proposition 20. The proof is trivial. (50) can be rewritten as

\[ \int_T^R e^{-\gamma z} f(z)dz = \frac{\kappa(\alpha - 1)}{\mu T}, \]  

(56)

where \( f(z) \equiv e^{-nz} (\alpha - e^{-\beta z}) > 0 \) if \( 0 < T \leq z \leq R < J \). Denoting the left-hand-side of (56) by \( I(\gamma) \) and differentiating with respect to \( \gamma \), one has

\[ \frac{dI(\gamma)}{d\gamma} = - \int_T^R e^{-\gamma z}zf(z)dz < 0, \]

if \( T < R \), i.e., \( I(\gamma) \) is strictly decreasing in \( \gamma \). Therefore, at most one unique \( \gamma \) exists which satisfies (56).

Proof of Proposition 21. From (1) and (51) this turns out to be trivial.
Acknowledgements

7 REFERENCES


Meltzer, D., 1995. Mortality decline, the demographic transition and economic growth. Brigham and Women’s Hospital and NBER, mimeo.


Reinhart, V. R., 1999. Death and taxes: their implications for endogenous


Tables

Table 1:

<table>
<thead>
<tr>
<th>Country</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$n$</th>
<th>$EV_O$</th>
<th>$EV_T$</th>
<th>$\hat{\alpha}_O$</th>
<th>$\hat{\alpha}_T$</th>
<th>$J$</th>
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<td>0.005</td>
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<td>78.9</td>
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<td>36.3</td>
<td>109.2</td>
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Table 2: Interior vs. Corner Solutions

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<tr>
<th>$0 &lt; \eta \leq \eta_0$</th>
<th>$\eta_0 &lt; \eta &lt; \eta^*$</th>
<th>$\eta \geq \eta^*$</th>
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</thead>
<tbody>
<tr>
<td>$0 = T = R &lt; J$</td>
<td>$0 &lt; T &lt; R &lt; J$</td>
<td>$T = T_{\text{max}}(\eta^<em>) = \frac{J_0}{\eta^</em>}$</td>
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<tr>
<td>$R = J$</td>
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</table>

Table 3. Benchmark case.

Parameter Values

- $n^{(b)} = 0.010$, $\beta = -0.017$, $\alpha = 7.706$, $\mu = 0.311$, $\eta_0 = 0.081$, $\phi = 0.450$, $\omega = 25.312$, $s^{(d)} = 0.153$, $\eta = 0.140$, $\eta_* = 0.216$. 

Main Results

- Maximum age, $J : 120.12$
- Median age$^{(c)}$, $\hat{\alpha}: 32.01$
- Per capita growth rate$^{(h)}$, $\gamma: 2.1\%$
- Replacement rate$^{(e)}$, $\theta(1 - s): 0.36$
- Education$^{(g)}$, $T : 35.81$
- Life expectancy$^{(o)}$, $EV : 79.21$
- Mean age$^{(c)}$, $\bar{\alpha}: 37.53$
- Fertility ratio, $1/\kappa : 0.019$
- Retirement age$^{(l)}$, $R : 61.79$

Key to Table 3


(d) The choice of \( s \) reproduces the social security tax rate, equally shared by employers and workers, as reported in Diamond and Gruber (1997), p. 9, and in Coronado, Fullerton and Glass (2000), p. 10.

(e) The resulting replacement rate \( \theta \) (defined on net wages) equals 0.43 (or, equivalently, 0.36 on gross wages). Observed replacement rates vary substantially depending on the worker’s individual characteristics (retirement age, birth year, marital status, average labor income along the active period, ...). According to the simulation that Diamond and Gruber (1997) run in their base case, a male individual born in January 1930, married, and with a labor income equal to average one among his generation would obtain a replacement rate 0.40 if retired at 62 (the minimum). And if he retired at 69, his replacement rate would be 0.90. [See Diamond and Gruber (1997), p. 20, and Table 2 for this base case, and pp. 22-24 and Tables 3-5 for the rest of cases considered.] Galasso (1999) reports an average replacement rate of 0.517 in 1999 and 0.61 in 1995 in the US.


(g) The value 35.81 is a long way from the observed mean value of 12.33 years [See Butcher and Case (1994), Blackburn and Neumark (1993).] \(^{28}\) As an immediate consequence, the return to an additional year of education or (per unit) increment in human capital is 0.028, reasonably close to the reported values in the empirical literature.

(h) The same growth rate is obtained in Devereux and Love (1994) and King and Rebelo (1990), slightly higher than 1.5% obtained in Lucas (1990). Data for per capita

\(^{28}\)For the sake of comparison, Bassanini and Scarpetta (2001), p. 28, show the average years of education among working population for 21 OECD countries between 1971 and 1978. Figures show an increasing trend in time. For instance, the USA average goes from 11.6 in 1971 to 12.7 in 1998. Among the countries in the sample, in 1998 Germany ranked first (13.5 years) and Portugal last (7.7).
GDP growth rate can be obtained in 2000 Statistical Abstract of the United States, U.S. Census Bureau.
Figures