Cheap Money and Risk Taking: 
Opacity versus Underlying Risk

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Abstract
In a Bayesian setting, investments can be risky either because they are opaque, i.e., their payoff-relevant signals are noisy, or because they are fundamentally risky, i.e., the variance of the prior is high. When interest rates are low (high), investors favor opaque (transparent) projects that are perceived to be fundamentally safe (risky). Therefore, whether low interest rates lead to increased risk taking depends on the sources of risk. Moreover, this analysis helps explain the popularity of senior tranches of CDOs in the pre-crisis years, which were characterized by an unusual combination of high opacity and, supposedly, low fundamental risk.

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1 Introduction

A consensus has emerged that interest rates were exceedingly low in the run-up to the recent financial crisis.\textsuperscript{1} This observation has led to questions about the implications of cheap money for financial stability and, more specifically, for the level of risk taking in the financial system.

A number of mechanisms are frequently invoked to explain higher risk taking in response to low interest rates. These mechanisms tend to rely on the observation that low interest rates boost asset values. As asset values rise, balance sheets of banks grow, their leverage declines, and their risk taking and lending capacity expands. The greater willingness to lend may result in more risk taking to the extent that the set of safe borrowers is more or less fixed (Adrian and Shin, 2009). For non-banks, a similar dynamic is at play. Higher asset values increase collateral values and, thereby, create opportunities for additional risk taking on the part of investors. The motive for additional risk taking is then provided by the fact that risk tolerance tends to rise with wealth, such that higher asset values also make investors want to take on more risk. (Borio and Zhu, 2008; Gambacorta, 2009).

While plausible, these theories do not specifically address a key aspect of the crisis, namely, that so much of the risk was concentrated in highly complex and, therefore, opaque financial instruments. Moreover, a salient feature of many of these instruments was that, despite their opacity, they were perceived to be fundamentally rather safe due to the way they were structured. This is particularly true for senior tranches of collateralized debt obligations (CDOs), which often held AAA credit ratings. The goal of this paper, then, is to provide an explanation for the extraordinary popularity of these kinds of highly opaque but (seemingly) safe financial instruments in the low interest rate environment leading up to the crisis.\textsuperscript{2}

We develop a simple Bayesian investment screening model in which we distinguish between underlying, or “fundamental,” risk on the one hand, and opacity risk on the other. Underlying risk corresponds to the variance of investors’ prior beliefs about the payoff distribution of investments in a certain asset class. Opacity risk corresponds to the noisiness of the payoff-relevant signal about a particular investment project in that class. While both sources of risk contribute to the overall riskiness of an investment project in essentially the same way, we show that changes in interest rates affect risk taking in these two types of risk very differently. When interest rates are high, investors tend to invest in transparent projects with high underlying risk. When interest rates are low, investors favor opaque but (seemingly) fundamentally safe investments, such as senior tranches of CDOs.

The fact that changes in interest rates affect opacity and underlying risk taking in opposite ways makes the net effect of low interest rates on overall risk taking fundamentally ambiguous. We show that the net effect (as measured by the average riskiness of financed projects) depends on whether the collection of potential investments differs mostly in terms of opacity or mostly in terms of underlying risk. If potential investments differ in terms of opacity but are relatively similar in terms of underlying risk, low interest rates increase risk

\textsuperscript{1}Some observers, such as Taylor (2009a, 2009b), fault monetary policy. Others, among them Bernanke (2005), blame a global savings glut.

\textsuperscript{2}New CDO issuance reached a peak of about $1.1 trillion in 2006. About half was based on actual bonds and loans, while the other half was synthetic, i.e., based on credit default swaps. Approximately 80% of the value of CDOs was rated AAA. (Baird, 2007.)
taking. This is consistent with the common narrative about the causes of the crisis. If, on the other hand, potential investments differ in terms of underlying risk but are relatively similar in terms of opacity, low interest rates decrease risk taking. This second effect of low interest rates goes against the received wisdom.

In one version of our model, we compare two classes of investment projects with the same underlying payoff risk but different degrees of opacity. That is, investors observe more informative signals about the future payoffs of “transparent” projects than about “opaque” projects. In the other version of the model, the informativeness of signals is the same for both classes of projects, but one class exhibits greater underlying payoff risk than the other. In both versions, investors are risk-neutral, their funding costs are given by an exogenous real gross interest rate \( R \), and they know the type of project they are dealing with—i.e., opaque versus transparent, or fundamentally safe versus fundamentally risky.

First, consider the version of the model with transparent and opaque projects that have the same underlying payoff risk. Investors will choose to invest in a project if and only if it has an expected payoff conditional on its signal greater than or equal to \( R \). We show that the threshold signals for transparent projects and those for opaque projects vary with \( R \) such that investors invest more in opaque projects when interest rates are low, and more in transparent projects when interest rates are high. The intuition for this result is as follows. Because transparent signals are more informative than opaque signals, a positive signal from a transparent project is better news than a similar signal from an opaque project—while a negative signal is worse news. In other words, investors update their expectations more aggressively in response to transparent signals than in response to opaque signals. For high interest rates, only projects that are sufficiently “upgraded” in response to the observed signal receive funding. Because of the higher sensitivity of the posterior, transparent projects have an easier time jumping that hurdle than opaque projects. Hence, for high interest rates, transparent projects are more likely to be financed than (otherwise similar) opaque projects. For low interest rates, by contrast, only projects that are sufficiently “downgraded” in response to their signal do not get funded. The higher sensitivity of the posterior now implies that transparent projects are more likely to be screened out and not receive financing than opaque projects.

We strengthen this result by establishing monotonicity: above some threshold funding rate \( R^* \), a fall in interest rates always increases the share of opaque investments in investors’ portfolios. When \( R \) falls below \( R^* \), there even exist strictly profitable opaque projects that have a better chance of financing than equally profitable transparent projects. Finally, we show that when expectations deteriorate during an economic downturn or crisis—modeled as a downward revision of the expectation of underlying payoffs—a “flight-to-quality” occurs: the value of opaque projects is marked down more sharply than the value of transparent projects.

In the second version of our model, the two types of projects generate equally informative signals but differ in underlying riskiness. As before, risk-neutral investors choose to invest in all projects with expected payoffs conditional on their signals greater than or equal to \( R \). The threshold signals for high-risk projects and for low-risk projects vary with \( R \) such that investors invest more in fundamentally safe projects when interest rates are low and more in fundamentally risky projects when interest rates are high—a relationship that is once again essentially monotone. The intuition is again based on the sensitivity of the posterior with
respect to the value of the observed signal. In this case, expectations for fundamentally risky projects are revised more aggressively (either up or down) than for fundamentally safe projects. Hence, at high interest rates, riskier projects have a greater chance of sufficient upgrading in order to get financing, while at low interest rates, they have a greater chance of sufficient downgrading such that they do not get financing.

Combining the two versions of our model, we find that the amounts of underlying risk and opacity risk taken on by investors tend to move in opposite directions in response to interest rate changes. At low interest rates, investors choose opaque but fundamentally safe projects, while at high interest rates they favor transparent but fundamentally risky projects. This makes the net effect of low interest rates on risk taking highly ambiguous and dependent on the specific characteristics of the underlying population from which investments are selected.

**Related Literature**

Technically, our paper is closely related to the literature on statistical discrimination (see, e.g., Phelps, 1972; Aigner and Cain, 1977; Cornell and Welch, 1996; Morgan and Várdy, 2009). Indeed, transparent and opaque investment projects are similar to “mainstream” and “diverse” job candidates in Morgan and Várdy (2009). In the operations research literature, Baker (2006) has studied the relationship between risk and informativeness, and derived an equivalence between the two. Even though her set-up and focus are rather different, the analyses are clearly related. More broadly, our paper is, of course, also closely related to the literature on the effect of interest rates on investor behavior and, in particular, risk taking. (See, e.g., Fishburn and Porter, 1976; Wong, 1997; Viaene and Zilcha, 1998; and, for a literature review, De Nicolò et al., 2010.) In the Capital Asset Pricing Model (Sharpe, 1964; and Lintner, 1965), the market portfolio of risky assets becomes less risky when the risk-free interest rate declines—even if, ultimately, risk-averse investors choose to hold less of the riskless asset and more of the market portfolio. A fall in interest rates also decreases risk in basic models of asymmetric information, because lower interest rates cause adverse selection to be less severe, which makes the loan applicant pool less risky (Stiglitz and Weiss, 1981).

In the wake of the recent financial crisis, the notion that low interest rates boost risk taking has become the more popular view (Borio and Zhu, 2008; and Jimenez et al., 2008). Indeed, Taylor (2009a, 2009b) argues that the financial crisis was a direct consequence of an excessively loose monetary policy. Finally, Dell’Ariccia and Marquez (2006) demonstrate that low interest rates reduce banks’ screening incentives and thus lead to riskier loan portfolios.

## 2 Model

Consider a collection of investment projects, \( I \). Each project \( i \in I \) requires an investment of 1 today and produces a (real) payoff \( q_i \) next period. A project’s payoff \( q_i \) is the realization of a random variable \( \tilde{q}_i \), which is Lognormally distributed. Specifically, \( \tilde{q}_i = e^{\tilde{\pi}_i} \), where \( \tilde{\pi}_i \) is Normally distributed with mean \( \mu \) and variance \( \sigma_0^2 \). If we let \( \nu \equiv \mu + \frac{1}{2} \sigma_0^2 \), then the distribution of \( \tilde{q}_i \) is Lognormal with mean \( e^\nu \) and standard deviation \( e^\nu \sqrt{e^{\sigma_0^2} - 1} \). We call the variability of \( \tilde{q}_i \) the “underlying” or “fundamental” payoff risk of investment project \( i \) and measure it by the normalized payoff variance \( \left( \frac{\text{Stddev}[\tilde{q}_i]}{E[\tilde{q}_i]} \right)^2 = e^{\sigma_0^2} - 1 \) or, for convenience,
simply by $\sigma_0^2$.

A project’s payoff is not observable at the time of investment. However, before the investment decision is made, each project is costlessly screened with an information technology that produces a payoff-relevant random signal, $\tilde{y}_i$. Specifically, a project’s signal, $\tilde{y}_i$, is equal to the true underlying $\pi_i$ plus white noise. That is,

$$\tilde{y}_i = \pi_i + \tilde{\varepsilon}_i$$

where $\tilde{\varepsilon}_i$ is Normally distributed with mean zero and variance $\sigma_1^2$, while $\tilde{\pi}_i$ and $\tilde{\varepsilon}_i$ are independent. We call the noisiness of $\tilde{\varepsilon}_i$ the “opacity” of project $i$ and measure it by variance $\sigma_1^2$. The foregoing implies that signal $\tilde{y}_i$ is Normally distributed, with mean $\mu$ and variance $\sigma_0^2 + \sigma_1^2$. We denote the cumulative distribution function (CDF) of $\tilde{y}_i$ by $F$ and its probability density function (PDF) by $f$. The posterior distribution of $\tilde{\pi}_i$ conditional on $y_i$ is Lognormal with mean $E[\tilde{\pi}_i | y_i] = e^{\sigma_1^2/2} - 1$ and standard deviation $E[\tilde{\pi}_i | y_i] \sqrt{e^{\sigma_1^2/2} - 1}$. To keep the inference problem manageable, we assume that investment projects are—or can be—evaluated separately. This is assured if projects in $I$ are screened by different investors—one for each project—or if $\tilde{y}_i$ and $\tilde{\pi}_j$ are independent for $i \neq j$.

We measure the “overall” riskiness of project $i$ by the normalized payoff variance of its posterior, $\left(\frac{\text{StdDev}[\tilde{\pi}_i | y_i]}{E[\tilde{\pi}_i | y_i]}\right)^2 = e^{\sigma_1^2/2} - 1$, or, for convenience, simply by $\sigma^2 \equiv \frac{\sigma_1^2 \sigma_1^2}{\sigma_0^2 + \sigma_1^2}$. Note that the two sources of risk, i.e., fundamental risk $\sigma_0^2$ and opacity $\sigma_1^2$, contribute symmetrically to the overall riskiness of a project. For future reference, let $\hat{\sigma}_0^2$, $\hat{\sigma}_1^2$, and $\hat{\sigma}^2$ denote the average underlying risk, opacity, and overall riskiness of projects in $I$, while $\hat{\sigma}_0^2$, $\hat{\sigma}_1^2$, and $\hat{\sigma}^2$ denote the average underlying risk, opacity, and overall riskiness of financed projects, i.e., of projects in investors’ portfolios.

Let $r$ denote the real riskless interest rate in the economy and $R = 1 + r$ the gross cost of financing a project. Investors are risk-neutral. Hence, a project is financed if and only if its expected value conditional on its signal, $E[\tilde{\pi}_i | y_i]$, is at least $R$.

In Section 3, we assume that all projects in $I$ have the same underlying risk, $\sigma_0^2$, but differ in opacity, $\sigma_1^2$. We analyze which projects are financed and how the share of opaque versus transparent projects in investors’ portfolios changes with the level of the interest rate. We also compare the average payoffs of financed opaque projects with those of financed transparent projects and study relative valuation changes in response to changes in economic outlook or investor sentiment. In Section 4, we analyze the polar opposite case where all projects are equally opaque—i.e., they have the same $\sigma_1^2$—but differ in their levels of underlying risk, $\sigma_0^2$. In Section 5, we allow projects to differ both in terms of opacity and in terms of underlying risk, and focus on the effect of interest rates on overall risk taking. Section 6 concludes. In the main text, we limit attention to equity financing. In Appendix A, we show that our results remain unchanged when investment projects are financed by debt, or by a mix of debt and equity. Finally, Appendix B contains proofs relegated from the main text.
3 Transparent versus Opaque Projects

In this section, we consider the case where all projects in $I$ have the same underlying risk but differ in their levels of opacity. Specifically, collection $I$ contains two types of projects $\kappa$, $\kappa \in \{P, T\}$, where $P$ denotes opaque projects and $T$ denotes transparent projects. Opaque projects have the same ex ante expectation $e^\nu$ and underlying risk $\sigma_0^2$ as transparent projects, but greater opacity $\sigma_1^2$. That is, type $P$ projects generate relatively uninformative signals $\tilde{y}_P$ with variance $\sigma_1^2_{1,P}$, while type $T$ projects generate relatively informative signals $\tilde{y}_T$ with variance $\sigma_1^2_{1,T} < \sigma_1^2_{1,P}$. Opaque projects make up a share $\alpha < 1$ of projects in $I$. The remaining projects in $I$, with share $1 - \alpha$, are transparent. At the time of investment, investors know a project’s type.

A profit maximizing risk-neutral equity investor finances a project of type $\kappa$ if and only if the project’s signal $y_\kappa$ is such that $E[\tilde{q}_\kappa | y_\kappa] \geq R$. Solving for $y_\kappa$, we get

$$y_\kappa \geq \ln R + \frac{\sigma_{1,\kappa}^2}{\sigma_0^2} (\ln R - \nu) \equiv y_\kappa^R$$

Hence, $y_\kappa^R$ denotes the threshold signal such that $E[\tilde{q}_\kappa | y_\kappa] = R$.

**Financing Probabilities** We begin by analyzing the effect of $R$ on the relative financing probabilities of opaque versus transparent projects. Then, we look at the implications for the relative representation of opaque versus transparent projects in investors’ portfolios; i.e., among projects that are actually financed. Finally, we determine the average opacity and overall riskiness of financed projects relative to the average opacity and overall riskiness of projects in $I$.

Denote the probability that a (random) project of type $\kappa$ is financed by $p_\kappa^R$. I.e., $p_\kappa^R \equiv \Pr(\tilde{y}_\kappa \geq y_\kappa^R)$. The probability ratio $p_P^R/p_T^R$ can be interpreted as the share of opaque versus transparent projects in investors’ portfolios relative to their underlying population shares $\alpha$ and $1 - \alpha$. If this probability ratio is equal to 1, the portfolio shares exactly match the population shares. Otherwise, one of the two types of projects is relatively overrepresented among financed projects. Define $\ln R^* \equiv \nu + \frac{1}{2} i_{\kappa} \sigma_T$, where $i_{\kappa} \equiv \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_{1,\kappa}^2}}$. Then,

**Proposition 1** The following statements are equivalent:\(^3\)

1. Interest rates are low. I.e., $R < R^*$.
2. A random opaque project has a better chance of financing than a random transparent project. I.e., $p_P^R > p_T^R$.
3. Opaque projects are overrepresented in investors’ portfolios. I.e., $p_P^R/p_T^R > 1$.
4. Financed projects are, on average, more opaque and overall riskier than projects in $I$. I.e., $\hat{\sigma}_1^2 > \hat{\sigma}_1^2$ and $\hat{\sigma}^2 > \hat{\sigma}^2$.

\(^3\)If interest rates are high, i.e., $R > R^*$, all inequalities are reversed.
The equivalence between statements 2, 3, and 4 in Proposition 1 are straightforward. The intuition for the equivalence between statements 1 and 2 relies on the fact that payoff expectations are revised more strongly in response to transparent signals than in response to opaque signals. For high interest rates, only those projects receive funding that, in response to their signal, are sufficiently upgraded relative to prior beliefs. Because signals from opaque projects are less informative than signals from transparent projects, beliefs about opaque projects tend to be revised less than beliefs about transparent projects. This explains why random opaque projects are less likely to be funded than random transparent projects when interest rates are high.

For low interest rates by contrast, in order to receive funding, projects must not disappoint too much. Hence, in this case, large (downward) belief revisions cause projects not to be funded. As before, opaque projects tend to experience smaller belief revisions than transparent projects. This explains why, for low interest rates, random opaque projects are more likely to be funded than random transparent projects.

We now expand on this intuition and offer a graphical interpretation of Proposition 1, which also allows us to better assess the robustness of our result. First, we transform the signals $\tilde{\eta}_\kappa$ through their CDFs into alternative but equivalent signals $\tilde{\zeta}_\kappa$. That is, define

$$\tilde{\zeta}_\kappa \equiv F_\kappa (\tilde{\eta}_\kappa)$$

The transformed signals for opaque and transparent projects, $\tilde{\zeta}_P$ and $\tilde{\zeta}_T$, are identically distributed—namely, uniformly on $[0,1]$. This makes them comparable in the following sense: in order to determine which type of project has a better chance of financing, it now suffices to check which of the transformed threshold signals, $\tilde{\zeta}_P^R$ or $\tilde{\zeta}_T^R$, is smaller. Specifically,

$$p_P^R \geq p_T^R \iff \tilde{\zeta}_P^R \leq \tilde{\zeta}_T^R$$

Moreover, because signal distributions are uniform, financing probabilities are simply equal to $p_\kappa^R = 1 - \tilde{\zeta}_\kappa^R$.

Figure 1 plots the conditional expectations $E[\tilde{\eta}_\kappa | z_\kappa]$, $\kappa \in \{P,T\}$, of opaque and transparent projects as a function of their normalized signals $z_P$ and $z_T$. In these graphs, for each $R$, $z_\kappa^R$ is easily identified as the signal $z_\kappa$ that makes $E[\tilde{\eta}_\kappa | z_\kappa] = R$. Note that the lower informativeness of $\tilde{\zeta}_P$ relative to $\tilde{\zeta}_T$ translates into $E[\tilde{\eta}_P | z_P]$ being “flatter” than $E[\tilde{\eta}_T | z_T]$. Hence, the former crosses the latter exactly once and from above at $R = R^*$, such that opaque projects have a better chance of financing than transparent projects if and only if $R \leq R^*$.

This graphical interpretation also shows that Proposition 1 is, in fact, quite robust and does not crucially depend on our specific distributional assumptions. Indeed, the result carries over to all payoff distributions and informativeness criteria that imply single-crossing of the (normalized) conditional expectation functions for opaque versus transparent information systems. (See Gauza and Penalva, 2010, for a discussion of “single-crossing informativeness.”)

**Profitable Projects** We now focus on the subset of projects that are in fact profitable; i.e., projects with $\eta_\kappa \geq R$. One might expect that strictly profitable projects that are transparent always have higher financing probabilities than equally profitable opaque projects.
Consider, for example, the (admittedly extreme) case where transparent signals are perfectly informative. In that case, all profitable transparent projects get funded, but only a fraction of profitable opaque projects. While this intuition seems appealing, the next proposition shows that it is wrong.

Denote by $p^R_\kappa(q)$ the probability that a project of type $\kappa$ with payoff $q$ is financed when the interest rate is $R$. Then,

**Proposition 2** When interest rates are very low, there exist strictly profitable opaque projects that have a better chance of financing than equally profitable transparent projects.

Formally, $\ln R < \nu \iff \exists q > R : p^R_\kappa(q) > p^R_T(q)$.

To understand the intuition behind Proposition 2, consider a project that just breaks even. That is, the project’s true payoff $q_\kappa$ is exactly equal to $R$, such that an investor would make neither a profit nor a loss. If signals are almost perfectly informative, then a break-even project has an approximately 50% chance of being financed, independent of $R$. If signals are almost uninformative, on the other hand, then the chances of financing do very much depend on $R$: when $R$ is high relative to a random project’s unconditional expected payoff $e^\nu$, then, for the usual reasons related to the sensitivity of the posterior with respect to the signal, a highly opaque break-even project has almost no chance of financing. Hence, for high interest rates, a break-even project clearly stands a better chance when it is transparent than when it is opaque. When $R$ is very low, the payoff expectation’s insensitivity to uninformative signals gives a break-even project almost guaranteed funding if it is opaque, while it still only has a 50/50 chance if it is highly transparent. Hence, now, a break-even project stands a better chance of being financed if it is opaque. Finally, by continuity, the same argument holds for strictly profitable projects with payoffs $q_\kappa$ greater than, but close, to $R$. 
Proposition 2 suggests that in low interest rate environments, investment banks selling strictly profitable investments may face incentives to artificially increase the opacity of assets, as long as they can convince investors that this opacity is unavoidable. For unprofitable “junk” projects, this would not be surprising. Indeed, cloaking junk in smoke may be the only way to get anybody to even consider it. It is more surprising is that a seller may want to muddy the water for perfectly good projects. Of course, in our model, there is no information asymmetry between the seller (the investment bank) and the buyer (the investor). Moreover, the seller has no control over the informativeness of a project’s signal. Nonetheless, it remains true that the range of payoffs for which transparency is actually advantageous for the seller becomes smaller as \( \omega \) falls. This means that developing complex and, thus, opaque financial instruments becomes more attractive, even if the bank knows no more about their payoffs than the investor.

**Monotonicity** Above, we compared financing probabilities of transparent and opaque projects under high and low interest rates. We found that investors favor transparent projects when interest rates are high, and opaque projects when interest rates are low. We now show that, above some threshold, the shift from transparent to opaque projects is, in fact, monotone in \( \omega \).

**Proposition 3** If \( \ln \omega \geq \nu \), then a marginal reduction in interest rates raises the share of opaque projects in investors’ portfolios.

Formally,

\[
\ln \omega \geq \nu \implies \frac{d \left( \frac{p^R_I}{p^R_T} \right)}{d \omega} < 0
\]

The monotonicity of \( \frac{p^R_I}{p^R_T} \) implies that, for \( \ln \omega \geq \nu \), a fall in interest rates always increases the average opacity—and, thus, the average overall riskiness—of financed projects. Figure 2, which depicts \( \frac{p^R_I}{p^R_T} \) as a function of \( \omega \), graphically illustrates both Propositions 1 and 3. Consistent with Proposition 1, the figure shows that for \( \omega > \omega^* \), \( \frac{p^R_I}{p^R_T} < 1 \), such that opaque projects are underrepresented relative to their population share. Conversely, for \( \omega < \omega^* \), \( \frac{p^R_I}{p^R_T} > 1 \), such that opaque projects are overrepresented. Consistent with Proposition 3, the figure also shows that for \( \ln \omega \geq \nu \), \( \frac{p^R_I}{p^R_T} \) is a monotonically decreasing function of \( \omega \). Therefore, in that range, a fall in \( \omega \) always increases the share of opaque projects in investors’ portfolios. Even when \( \ln \omega \) falls below \( \nu \) (i.e., \( \omega \) falls below 1 in Figure 2), initially, the probability ratio \( \frac{p^R_I}{p^R_T} \) remains decreasing in \( \omega \). Note, however, that when \( \omega \downarrow 0 \), all projects in \( I \) are financed, irrespective of their type and signal. That is, \( \lim_{\omega \downarrow 0} \left( \frac{p^R_I}{p^R_T} \right) = 1 \). By continuity, there must then be a “turning point” \( \omega' \), \( 0 < \omega' < \omega^* \), where \( \frac{p^R_I}{p^R_T} \) becomes upward sloping for \( \omega < \omega' \). In other words, \( \frac{p^R_I}{p^R_T} \) cannot be globally decreasing.

**Average Payoffs of Financed Projects** How does the average payoff of a financed transparent project compare with that of a financed opaque project? The answer to that question might seem obvious. Indeed, the fact that transparent and opaque signals can be ordered according to Blackwell’s (1953) informativeness criterion implies that investors always prefer transparent signals over opaque ones. That is, investors’ overall profits are
Under and Overrepresentation of Opaque versus Transparent Projects

Figure 2: The ratio of financing probabilities as a function of the gross interest rate, R. Opaque projects are overrepresented in investors’ portfolios if and only if $R < R^*$. (Parameter values: $\sigma_0 = 1$, $\sigma_{1,T} = 1$, $\sigma_{1,P} = 2$, $\nu = 0$)

increasing in the degree of signal transparency. However, from the fact that overall profits are greater, one may not conclude that the average payoff of a financed transparent project is necessarily greater than that of a financed opaque project. For high interest rates, much fewer opaque than transparent project are undertaken. Hence, the average profit of a financed opaque project could very well be higher than that of a financed transparent project, while ex ante average profits from a random opaque project remain lower.

Nonetheless, the next Proposition shows that for Lognormally distributed payoffs, even though it is not implied by Blackwell (1953), the average payoff of financed transparent projects is always larger than that of financed opaque projects.

**Proposition 4** For all $R$, the average payoff of transparent projects in investors’ portfolios is strictly greater than that of opaque projects.

**Change in Expectations** Finally, we study what happens to the market value (i.e., expected return) of existing investments when investor sentiment changes. For instance, investors might conclude that their previous prior beliefs were too optimistic given the current economic environment. How will this downward revision of expectations affect the value of opaque projects relative to that of transparent projects? We model a change in investor sentiment by a change in $\nu$. We say that investors become more optimistic if $\nu$ rises and more pessimistic if $\nu$ falls. Let $\gamma_{\nu} = \frac{dE[q_\nu|y_\nu]}{d\nu} \frac{1}{E[q_\nu|y_\nu]}$. That is, $\gamma_{\nu}$ is the rate of change of the conditional expected return when $\nu$ increases. Note that $\gamma_{\nu} = \frac{\sigma_{2,\nu}}{\sigma_0^2 + \sigma_{1,\nu}^2}$. Hence,

**Proposition 5** The value of opaque projects is uniformly more sensitive to changes in investor sentiment than the value of transparent projects.

Formally, for all $y_P$ and $y_T$, $\gamma_P > \gamma_T$. 

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The intuition for Proposition 5 is straightforward. Because payoffs are Lognormally distributed, the expectation of the posterior, $E[\tilde{q}_k \mid y_k]$, is a monotone function of a convex combination of $\nu$ and the signal $y_k$. The less informative the signal, the more weight is put on $\nu$ and the less on $y_k$. Hence, opaque projects are more affected by revisions of $\nu$ than transparent projects.

4 High versus Low Underlying Risk

We now consider the case where all projects in $I$ are equally opaque, but differ in their underlying risk. That is, collection $I$ contains two types of projects $\lambda$, $\lambda \in \{H, L\}$, where $H$ denotes fundamentally risky projects and $L$ denotes fundamentally safe projects. Risky projects have the same opacity $\sigma^2_1$ as safe projects, but $\sigma^2_{0,H} > \sigma^2_{0,L}$. To ensure that both types of projects are equally attractive in ex ante terms, we assume that $(\mu_H, \mu_L)$ satisfy

$$\mu_H + \frac{1}{2}\sigma^2_{0,H} = \mu_L + \frac{1}{2}\sigma^2_{0,L} \equiv \nu$$

such that risky and safe projects have the same unconditional expected payoff, $e^\nu$. Fundamentally risky projects make up a share $0 < \beta < 1$ of projects in $I$. The remaining projects are fundamentally safe.

The analysis in this section is parallel to that in Section 3. The conclusions, however, are diametrically opposed. In Section 3 we saw that, ceteris paribus, low interest rates favored opaque and, thus, overall risky projects over transparent and, thus, overall safe projects. We now show that, at the same time, low interest rates favor fundamentally safe and, thus, overall safe projects over fundamentally risky and, thus, overall risky projects. As we discuss in detail in Section 5, this makes the net effect of interest rates on overall risk taking highly ambiguous.

A risk-neutral equity investor finances a project of type $\lambda$ if and only if the project’s signal $y_\lambda$ is such that $E[\tilde{q}_\lambda \mid y_\lambda] \geq R$. Solving for $y_\lambda$, we get

$$y_\lambda \geq \ln R + \frac{\sigma^2_1}{\sigma^2_{0,\lambda}} (\ln R - \nu) \equiv y^R_\lambda$$

**Financing Probabilities** Once again, we begin by deriving an equivalence between the level of interest rates and, in this case, (i) the financing probabilities of fundamentally safe versus fundamentally risky projects, (ii) the overrepresentation of fundamentally risky projects in investors’ portfolios, and (iii) the average fundamental riskiness and overall riskiness of financed projects. Define $\ln R^* \equiv \nu + \frac{1}{2}i_Hi_L$, where $i_\lambda = \frac{\sigma^2_{0,\lambda}}{\sqrt{\sigma^2_{0,\lambda} + \sigma^2_1}}$. Then,

**Proposition 6** The following statements are equivalent.\(^4\)

1. Interest rates are high. I.e., $R > R^*$.\(^{4}\)

\(^4\)If interest rates are low, i.e., $R < R^*$, all inequalities are reversed.
Figure 3: Conditional expectations of payoffs for fundamentally risky and fundamentally safe projects as a function of normalized signals $z$.

2. A random fundamentally risky project has a better chance of financing than a random fundamentally safe project. I.e., $p_H^R > p_L^R$.

3. Fundamentally risky projects are overrepresented in investors’ portfolios. I.e., $p_H^R / p_L^R > 1$.

4. Financed projects are, on average, fundamentally riskier and overall riskier than projects in I. I.e., $\hat{\sigma}_H^2 > \hat{\sigma}_L^2$ and $\hat{\sigma}^2 > \hat{\sigma}_L^2$.

The equivalence between statements 2, 3, and 4 in Proposition 6 is again straightforward. The equivalence between statements 1 and 2 is a consequence of single-crossing of the (normalized) conditional expectation functions $E[\tilde{q}_H | z_H]$ and $E[\tilde{q}_L | z_L]$, which is illustrated in Figure 3.

In Section 3, the flatness of the conditional expectation function of opaque projects relative to transparent projects implied that, at low interest rates, overall riskier projects had greater financing probabilities than overall safer projects. Conversely, at high interest rates, the financing probabilities of overall safer projects were greater. In the current setup, these rankings are reversed. The flatness of the conditional expectation function of fundamentally safe projects implies that, now, it is the overall safer projects that have greater financing probabilities at low interest rates, while, for high interest rates, the financing probabilities of overall riskier projects are greater. Hence, the effect of interest rates on the average riskiness of financed projects crucially depends on the source of risk. If differences in overall risk among projects are due mainly to differences in opacity, then low interest rates lead to more risk taking. On the other hand, if differences in overall riskiness stem mostly from differences in underlying risk, then low interest rates lead to less risk taking. We discuss this issue in more detail in Section 5.
Projects with Identical Payoffs  For opaque versus transparent projects, $p^R_H(q) - p^R_T(q)$ satisfies single-crossing in $q$. This means that low-$q$ projects have a better chance of financing if they are opaque rather than transparent, while high-$q$ projects have a better chance of financing if they are transparent rather than opaque. The cross-over point $q^\ast$ is such that, for $\ln R < \nu$, $q^\ast > R$. In words: for low interest rates, there are strictly profitable opaque projects that have a better chance of financing than equally profitable transparent projects. (Proposition 2.)

For fundamentally risky versus fundamentally safe projects, $p^R_H(q) - p^R_L(q)$ turns out not to depend on $q$. This means that, depending on $R$, either $p^R_H(q) \geq p^R_L(q)$ for all $q$, or $p^R_H(q) < p^R_L(q)$, also for all $q$. Formulated differently, if, for some $q$, a fundamentally risky projects with payoff $q$ has a greater probability of financing than a fundamentally safe project with the same payoff, then this ordering holds for all $q$.

Proposition 7 Fundamentally risky projects with payoff $q$ have a better chance of financing than fundamentally safe projects with the same payoff if and only if $\ln R > \nu$.

Formally, for all $q$, $p^R_H(q) > p^R_L(q) \iff \ln R > \nu$

Because the ranking of $p^R_H(q)$ and $p^R_L(q)$ in Proposition 7 holds for all $q$, it trivially also holds for projects with $q \geq R$, i.e., profitable projects. Therefore, at high interest rates, all profitable fundamentally risky projects have a better chance of financing than equally profitable fundamentally safe projects, while at low interest rates the opposite holds.

Monotonicity  As was the case for opaque versus transparent projects, we now show that, above some threshold, the shift from fundamentally risky to fundamentally safe projects in response to changes in the interest rate is monotone.

Proposition 8 If $\ln R \geq \nu$, then a marginal reduction in interest rates lowers the share of fundamentally risky projects in investors’ portfolios.

Formally,

$$\ln R \geq \nu \implies \frac{d(p^R_H/p^R_L)}{dR} > 0$$

Proposition 8 implies that, for $\ln R \geq \nu$, a fall in interest rates always decreases the average underlying riskiness—and, thus, average overall riskiness—of financed projects. The proposition is illustrated in Figure 4, which depicts $p^R_H/p^R_L$ as a function of $R$.

Average Payoffs of Financed Projects  In Section 3, we saw that financed transparent (and, thus, overall safer) projects have higher average payoffs than financed opaque (and, thus, overall riskier) projects. Consistent with the pattern of reversal uncovered so far, we now show that financed fundamentally risky projects have higher average payoffs than financed fundamentally safe projects. The intuition is that projects with higher underlying payoff risk have fatter right tails, which increase their expected payoff conditional on exceeding the threshold signal. In other words, the greater upside dominates the greater downside, which is mitigated by the screening procedure. Formally,

Proposition 9 For all $R$, the average payoff of fundamentally risky projects in investors’ portfolios is strictly greater than that of fundamentally safe projects.
Figure 4: The ratio of financing probabilities as a function of the gross interest rate, $R$. Fundamentally safe projects are overrepresented in investors’ portfolios if and only if $R < R^{**}$. (Parameter values: $\sigma_1 = 2$, $\sigma_H = 1.5$, $\sigma_L = 1$, $\nu = 0$.)

**Change in Expectations** Finally, how does a change in investor sentiment, as measured by $\nu$, affect the market value of fundamentally safe projects relative to fundamentally risky projects? Because $\gamma_\lambda \equiv \frac{dE[\tilde{q}_\lambda | y_\lambda]}{dy_\lambda} = \frac{\sigma_1^2}{\sigma_{0,\lambda}^2 + \sigma_1^2}$, we obtain the following result.

**Proposition 10** The value of fundamentally safe projects is uniformly more sensitive to changes in investor sentiment than the value of fundamentally risky projects.

Formally, for all $y_H$ and $y_L$, $\gamma_H < \gamma_L$.

The intuition is again simple. The safer the project, the more weight is put on $\nu$ and the less on $y_\lambda$ when calculating $E[\tilde{q}_\lambda | y_\lambda]$. Hence, the value of fundamentally safe projects is more sensitive to revisions of $\nu$ than the value of fundamentally risky projects.

## 5 Integrated Model

We now allow projects in $I$ to differ both in terms of opacity and in terms of underlying risk. Each project is characterized by (known) prior and signal variances $(\sigma_0^2, \sigma_1^2) \in [\bar{\sigma}_0^2, \bar{\sigma}_1^2]$, where $0 < \sigma_0^2 < \bar{\sigma}_0^2 < \infty$ and $0 < \sigma_1^2 < \bar{\sigma}_1^2 < \infty$. Projects are distributed on $[\bar{\sigma}_0^2, \bar{\sigma}_1^2] \times [\bar{\sigma}_0^2, \bar{\sigma}_1^2]$ according to some PDF $g(\sigma_0^2, \sigma_1^2)$. Denote the PDF of the variances of financed projects by $h$, and note that by Bayes’ rule,

$$h(\sigma_0^2, \sigma_1^2) = \frac{p^R(\sigma_0^2, \sigma_1^2) g(\sigma_0^2, \sigma_1^2)}{\int_{\sigma_1^2}^{\bar{\sigma}_0^2} \int_{\sigma_0^2}^{\bar{\sigma}_1^2} p^R(\sigma_0^2, \sigma_1^2) g(\sigma_0^2, \sigma_1^2) d\sigma_0^2 d\sigma_1^2}$$

As before, all projects have the same unconditional expectation, $e^\nu$, such that, in ex ante terms, they are equally attractive to risk-neutral investors. This implies that $\mu = \nu - \frac{1}{2} \sigma_0^2$.  

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Iso-Risk, Iso-Probability, and Iso-Sensitivity Curves  We begin by determining “iso-risk curves,” i.e., collections of projects that are equally risky overall. Two projects have the same overall risk if and only if they have the same normalized conditional variances

\[ \frac{\text{Var}(q|y)}{E[q|y]} = c\sigma^2 - 1. \]

As this expression depends on \((\sigma_0^2, \sigma_1^2)\) solely through \(\sigma^2 = \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 + \sigma_1^2}\), iso-risk curves in the \((\sigma_0^2, \sigma_1^2)\)-plane are described by \(\frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 + \sigma_1^2} = c\), where \(c\) is some positive constant (left panel of Figure 5). For simplicity, from here on, we will use \(\sigma^2\) as our measure of overall risk, rather than \(e^{\sigma^2} - 1\).

Next, we identify “iso-probability curves,” i.e., classes of projects that have the same financing probabilities, \(p^R\). Let \(i \equiv \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}}\) and denote the CDF and PDF of the standard Normal distribution by \(\Phi\) and \(\phi\), respectively. Recall that a project \((\sigma_0^2, \sigma_1^2)\) is financed if and only if \(E[q|z] \geq R\). This is equivalent to

\[ z \geq \Phi \left( \frac{1}{i} (\ln R - \nu) + \frac{1}{2i} \right) \]

where, by construction, \(z\) is uniformly distributed. Hence, the financing probability \(p^R\) of project \((\sigma_0^2, \sigma_1^2)\) is

\[ p^R (\sigma_0^2, \sigma_1^2) = 1 - \Phi \left( \frac{1}{i} (\ln R - \nu) + \frac{1}{2i} \right) \]

Financing probabilities \(p^R\) depend on \((\sigma_0^2, \sigma_1^2)\) solely through \(i\). Therefore, for all \(R\), projects with the same value for \(i\) have the same financing probability and, thus, lie on the same iso-probability curve. Intuitively, this result follows from the fact that all projects with the same \(i\) have identical conditional expectation functions, \(E[q|z]\).

Note that \(i\) is increasing in \(\sigma_0^2\) and decreasing in \(\sigma_1^2\). This makes iso-probability curves upward sloping in the \((\sigma_0^2, \sigma_1^2)\)-plane. To see why, suppose that we raise \(\sigma_0^2\) and thus steepen
the $E[\hat{q}|z]$-curve. To get back onto the initial iso-probability curve, we have to change $\sigma_1^2$ such that $E[\hat{q}|z]$ becomes flatter again and, thus, takes on its original steepness. This flattening of $E[\hat{q}|z]$ is achieved by also raising $\sigma_1^2$, which means that iso-probability curves are indeed upward sloping.

Finally, we consider how a change in $\sigma$ differentially changes the financing probabilities $p^R$ of the various projects $(\sigma_0^2, \sigma_1^2)$. As the derivative $\frac{dp^R(\sigma_0^2, \sigma_1^2)}{dR} = -\frac{1}{R} \phi \left( i \left( \ln R - \nu \right) + \frac{1}{2} i^2 \right)$ also depends on $(\sigma_0^2, \sigma_1^2)$ solely through $i$, iso-sensitivity curves and iso-probability curves coincide. Iso-sensitivity (and iso-probability) curves are drawn in the right panel of Figure 5.

Comparing iso-risk curves with iso-sensitivity curves, we see that the former are strictly downward sloping, while the latter are strictly upward sloping. Hence, they intersect at most once. This implies that two projects with the same overall riskiness (but different opacity and underlying risk) always have different interest rate sensitivities. Conversely, projects with the same interest rate sensitivity (but different opacity and underlying risk) always differ in their overall riskiness. Therefore, the effect of a change in interest rates on overall risk taking—as measured by the average riskiness of financed projects—can never be determined from merely knowing the overall riskiness of projects in $I$. For all projects, one also needs to know the sources of risk, namely, the levels of opacity and underlying risk. We return to this issue below.

**Varying $(\sigma_0^2, \sigma_1^2)$ for fixed $R$** In Sections 3 and 4, we studied the effect of interest rate changes on the (relative) financing probabilities of the various types of projects, and on the average riskiness of financed projects relative to projects in $I$. Before returning to these questions in the context of the integrated model, we first do the reverse exercise and, keeping $R$ fixed, study what happens to financing probabilities when $\sigma_0^2$ and $\sigma_1^2$ change.

Let $\bar{i} \equiv \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}}$ and $\hat{i} \equiv \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}}$, and note that $0 < \hat{i} < i < \bar{i} < \bar{\sigma}_0$.

**Proposition 11** For low interest rates, a marginal increase in opacity or a marginal decrease in underlying risk improve a project’s chances of financing. For high interest rates, the converse holds.

Formally,

$$\frac{\partial p^R}{\partial i} < 0 \quad \text{if} \quad \ln R < \nu + \frac{1}{2} \bar{i}^2$$

$$\frac{\partial p^R}{\partial i} > 0 \quad \text{if} \quad \ln R > \nu + \frac{1}{2} \bar{i}^2$$

Proposition 11 is an immediate consequence of the following lemma.

**Lemma 1** The effect of a marginal change in $i$ on a project’s financing probability satisfies

$$\frac{\partial p^R (>)}{\partial i} < 0 \iff i > \sqrt{2(\ln R - \nu)}$$

Proposition 11, which is illustrated in Figure 6, is broadly in line with the earlier observation that low interest rates favor opaque but fundamentally safe projects, while high interest rates favor transparent but fundamentally risky projects. The underlying intuition again relies on the opposing effects of $\sigma_0^2$ and $\sigma_1^2$ on the slope of the conditional expectation.
assumptions in the asset pricing literature, we stick to the Lognormal case throughout. See the Conclusion for a discussion of the robustness of our results to other payoffs.

Figure 6: Iso-probability curves for different levels of $R$. For low $R$, $p^R$ increases when we move to the northwest (left panel). For high $R$, $p^R$ increases when we move to the southeast (right panel). For unambiguously low interest rates, the $\ln$ function is generally simpler. However, in order to avoid negative values for $q$, and stay closer to the standard assumptions in the asset pricing literature, we stick to the Lognormal case throughout. See the Conclusion for a discussion of the robustness of our results to other payoff and signal distributions.

Function $E[\tilde{q}|x]$. Note, however, that this simple intuition only holds at the extremes of high and low interest rates. The reason is that, for intermediate values of $R$, what effectively constitutes "high" and "low" interest rates depends on $i$ and, hence, on $(\sigma_0^2, \sigma_1^2)$. Indeed, Lemma 1 implies that, for $R$ and $(\sigma_0^2, \sigma_1^2)$ such that $\nu + \frac{1}{2}i^2 < \ln R < \nu + \frac{1}{2}i^2$, the effects of marginal changes in $\sigma_0^2$ and $\sigma_1^2$ go in the same direction as under (unambiguously) low interest rates, i.e., $\ln R < \nu + \frac{1}{2}i^2$. Here, increasing $\sigma_0^2$ lowers a project’s chance of financing while increasing $\sigma_1^2$ raises it. For $R$ and $(\sigma_0^2, \sigma_1^2)$ such that $\nu + \frac{1}{2}i^2 < \ln R < \nu + i^2$, the effects of marginal changes in $\sigma_0^2$ and $\sigma_1^2$ go in the same direction as under (unambiguously) high interest rates, i.e., $\ln R > \nu + \frac{1}{2}i^2$. Here, increasing $\sigma_0^2$ raises a project’s chance of financing while increasing $\sigma_1^2$ lowers it. This implies that, for intermediate interest rates, the financing probability $p^R$ is single-peaked in $i$. In other words, there exists a locus of interior $\sigma_0^2$ and $\sigma_1^2$ that maximize a random project’s financing probability. The maximum occurs at the transition point between effectively “high” and “low” interest rates; i.e., where $i^2 = 2(\ln R - \nu)$. (Figure 6, center panel.) For unambiguously low interest rates, $p^R$ is maximized at the northwestern boundary point $(\sigma_0^2, \sigma_1^2)$. (Figure 6, left panel.) Finally, for unambiguously high interest rates, $p^R$ is maximized at the southeastern boundary point $(\sigma_0^2, \sigma_1^2)$. (Figure 6, right panel.)

Effect of $R$ on risk taking Finally, we return to the question of the effect of $R$ on overall risk taking. In Section 3, we considered projects with the same underlying risk and

\[ \text{In a model with Normally distributed payoffs, i.e., } q \sim N(\mu, \sigma_0^2), \text{ this complication would not arise. In that case, higher opacity (or lower underlying risk) raise a project’s financing probability if and only if } R < \mu. \text{ Indeed, all results in the paper carry over to the case of Normally distributed payoffs, while the algebra is generally simpler. However, in order to avoid negative values for } q, \text{ and stay closer to the standard assumptions in the asset pricing literature, we stick to the Lognormal case throughout. See the Conclusion for a discussion of the robustness of our results to other payoff and signal distributions.} \]
studied the effect of interest rates on the average opacity of financed projects. We showed that financed projects are on average more opaque—and, hence, overall riskier—than projects in \( I \), if and only if interest rates are low. Then, in Section 4, we considered projects with the same opacity and studied the effect of interest rates on the average underlying riskiness of financed projects. There, we showed that financed projects are on average fundamentally riskier—and, hence, overall riskier—than projects in \( I \), if and only if interest rates are high.\(^6\) We now analyze how these results carry over to the current bivariate setting, where projects in \( I \) differ both in terms of opacity and in terms of fundamental risk.\(^7\)

From Proposition 11 we know that, outside an intermediate range of interest rates, the sign of \( \frac{\partial p^R}{\partial \bar{i}} \) is unambiguous for all levels of opacity and underlying risk. We use this to show that:

**Proposition 12** If \( \bar{\sigma}_0^2 \) and \( \bar{\sigma}_1^2 \) are independent, then, for low interest rates, financed projects are on average more opaque but fundamentally less risky than projects in \( I \). For high interest rates, financed projects are on average less opaque but fundamentally riskier than projects in \( I \).

Formally, if \( \bar{\sigma}_0^2 \perp \bar{\sigma}_1^2 \), then

\[
\ln R < \nu + \frac{1}{2} \bar{i}^2 \implies \bar{\sigma}_1^2 > \bar{\sigma}_0^2 \text{ and } \bar{\sigma}_0^2 < \bar{\sigma}_1^2
\]

\[
\ln R > \nu + \frac{1}{2} \bar{i}^2 \implies \bar{\sigma}_1^2 < \bar{\sigma}_0^2 \text{ and } \bar{\sigma}_0^2 > \bar{\sigma}_1^2
\]

Comparing Proposition 12 to statement 4 of Propositions 1 and 6, what is perhaps most noteworthy is that Proposition 12 does not hold for arbitrary distributions of \( \bar{\sigma}_0^2 \) and \( \bar{\sigma}_1^2 \). To see that the claim in the proposition may fail when \( \bar{\sigma}_0^2 \) and \( \bar{\sigma}_1^2 \) are not independent, suppose that all projects in \( I \) lie on the same iso-probability curve. In that case, all (positively correlated) projects have the same financing probability and, therefore, the average opacity and underlying risk of financed projects is always the same as the average opacity and underlying risk of projects in \( I \). That is, for all \( R \), \( \bar{\sigma}_1^2 = \bar{\sigma}_0^2 \) and \( \bar{\sigma}_0^2 = \bar{\sigma}_1^2 \). Hence, Proposition 12 fails.

To complicate matters a bit, now suppose that all projects in \( I \) lie on a line in \( (\bar{\sigma}_0^2, \bar{\sigma}_1^2) \)-space that is uniformly steeper than the iso-probability curves in \([\bar{\sigma}_0^2, \bar{\sigma}_1^2]) \times [\bar{\sigma}_0^2, \bar{\sigma}_1^2] \). This means that an increase in \( \bar{\sigma}_0^2 \) is associated with such a large rise in \( \bar{\sigma}_1^2 \) that \( i = \frac{\bar{\sigma}_0^2}{\sqrt{\bar{\sigma}_0^2 + \bar{\sigma}_1^2}} \) actually declines. For this example, let us first compare \( \bar{\sigma}_0^2 \) to \( \bar{\sigma}_1^2 \) for high and low interest rates. Due to the negative correlation between \( \bar{i} \) and \( \bar{\sigma}_0^2 \), Proposition 11 implies that, for low interest rates, projects with above-average underlying risk have, in fact, above-average financing

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\(^6\)For ease of exposition, in Sections 3 and 4, we limited attention to binary project types; i.e., transparent versus opaque projects in Section 3, and fundamentally safe versus fundamentally risky projects in Section 4. Within the univariate settings of these sections, however, the monotone likelihood ratio properties of Propositions 3 and 8 allow the analyses to be readily extended to a continuum of types, without changing the results in an essential way.

\(^7\)In the univariate settings of Sections 3 and 4, we also derived monotonicity properties. (Propositions 3 and 8.) While there do exist counterparts for the bivariate setting, these results are not very appealing, because they only hold for “extreme” values of \( R \). Therefore, we have chosen to omit them from the paper.
probabilities. The above-average financing probabilities of projects with above-average underlying risk make financed projects on average fundamentally riskier than projects in \( I \). For high interest rates, by contrast, below-average financing probabilities of projects with above-average underlying risk make financed projects on average fundamentally less risky than projects in \( I \). Hence, in a partial reversal of Proposition \( 12, \ln R < \nu + \frac{1}{2} \mu^2 \implies \hat{\sigma}_0^2 > \hat{\sigma}_1^2 \), and \( \ln R > \nu + \frac{1}{2} \mu^2 \implies \hat{\sigma}_0^2 < \hat{\sigma}_1^2 \).

Second, we compare \( \hat{\sigma}_1^2 \) to \( \hat{\sigma}_1^2 \). In this example, an increase in \( \sigma_1^2 \) is associated with a sufficiently small increase in \( \sigma_0^2 \) that, unlike for average underlying risk, no reversal of comparative statics occurs with respect to average opacity. That is, just as in Proposition \( 12, \ln R < \nu + \frac{1}{2} \mu^2 \implies \hat{\sigma}_1^2 > \hat{\sigma}_1^2 \) and \( \ln R > \nu + \frac{1}{2} \mu^2 \implies \hat{\sigma}_1^2 < \hat{\sigma}_1^2 \). Combining the results for average underlying risk and average opacity, we find that, for low interest rates, financed projects are on average both more opaque and fundamentally riskier than projects in \( I \), while for high interest rates, financed projects are on average both less opaque and fundamentally less risky than projects in \( I \). That is, \( \ln R < \nu + \frac{1}{2} \mu^2 \implies (\hat{\sigma}_1^2 > \hat{\sigma}_1^2) \land (\hat{\sigma}_0^2 > \hat{\sigma}_0^2) \), while \( \ln R > \nu + \frac{1}{2} \mu^2 \implies (\hat{\sigma}_1^2 < \hat{\sigma}_1^2) \land (\hat{\sigma}_0^2 < \hat{\sigma}_0^2) \).

Analogous arguments show that if projects in \( I \) lie on an upward sloping line in \( (\sigma_0^2, \sigma_1^2) \)-space that is uniformly flatter than the iso-probability curves in \( [\sigma_0^2, \sigma_0^2] \times [\sigma_1^2, \sigma_1^2] \), then, for low interest rates, financed projects are on average both less opaque and fundamentally less risky than projects in \( I \), while for high interest rates, financed projects are on average more opaque and fundamentally riskier than projects in \( I \). That is, \( \ln R < \nu + \frac{1}{2} \mu^2 \implies (\hat{\sigma}_1^2 < \hat{\sigma}_1^2) \land (\hat{\sigma}_0^2 > \hat{\sigma}_0^2) \), while \( \ln R > \nu + \frac{1}{2} \mu^2 \implies (\hat{\sigma}_1^2 > \hat{\sigma}_1^2) \land (\hat{\sigma}_0^2 > \hat{\sigma}_0^2) \).

Finally, it is worth pointing out that independence between \( \sigma_0^2 \) and \( \sigma_1^2 \) is a sufficient but by no means necessary condition for the result in Proposition \( 12 \) to hold. In particular, the claim in the proposition will also hold for many (but not all) constellations where \( \hat{\sigma}_0^2 \) and \( \hat{\sigma}_1^2 \) are negatively correlated.\(^8\)

In light of the above, what do we conclude about the effect of low interest rates on risk taking? Do low interest rates cause overall risk taking to go up or down? Clearly, it can go either way. We just saw that when projects in \( I \) lie on a line in \( (\sigma_0^2, \sigma_1^2) \)-space that is steeper than the iso-probability curves, then low interest rates raise average opacity and underlying risk of financed projects, such that overall risk taking goes up for sure. On the other hand, when this line is flatter than the iso-probability curves (but still upward sloping), then low interest rates reduce average opacity and underlying risk of financed projects, such that overall risk taking goes down for sure.

If we are willing to assume that \( \hat{\sigma}_0^2 \) and \( \hat{\sigma}_1^2 \) are independently distributed then, at least, opacity risk and underlying risk do respond in a predictable way to low interest rates: the former goes up, while the latter goes down. The net effect on overall risk taking, however, still depends on whether increased opacity dominates or is dominated by reduced underlying risk. In turn, this depends on the particulars of the distributions of \( \hat{\sigma}_0^2 \) and \( \hat{\sigma}_1^2 \).

Finally, irrespective of the correlation structure between opacity and underlying risk, we can observe the following: when projects in \( I \) differ predominantly in terms of opacity rather than in terms of underlying risk then, essentially, the analysis in Section 3 applies. There,

\(^8\)Negative correlation does not guarantee the result in Proposition \( 12 \), however, because the sign of the correlation between two random variables is not invariant under monotone transformations of these random variables. This means, for example, that even if \( \hat{\sigma}_0^2 \) and \( \hat{\sigma}_1^2 \) are negatively correlated and \( \frac{\partial \bar{p}^R}{\partial \sigma_0} < 0 \), then \( \hat{\sigma}_1^2 \) and \( \bar{p}^R \) are not guaranteed to be positively correlated.
we saw that a fall in interest rates raises overall risk taking. If, on the other hand, projects in \( I \) differ predominantly in terms of underlying risk rather than in terms of opacity, then the analysis in Section 4 applies. In that case, a fall in interest rates lowers overall risk taking.

### 6 Conclusion

We have shown that, at low interest rates, risk-neutral investors favor opaque investment projects that are nonetheless perceived to be fundamentally safe. At high interest rates, investors are drawn to transparent projects with high underlying risk. We have argued that this may help explain the extraordinary popularity of certain kinds of complex financial instruments, such as senior tranches of CDOs, in the low interest rate environment of the pre-crisis years. At a fundamental level, our results follow from the observation that more precise information and more underlying risk have the same effect on the conditional expectation functions on which investment decisions of risk-neutral investors are based. Both factors make the conditional expectation functions steeper, i.e., more sensitive to payoff-relevant signals. The fact that greater underlying risk and greater informativeness have similar effects on investment decisions may seem paradoxical, since the former increases risk, while the latter decreases risk.

We have also shown that the effect of low interest rates on overall risk taking is highly ambiguous. If potential investments differ mostly in terms of opacity, then lower interest rates increase risk taking. If potential investments differ mostly in terms of underlying risk, then lower interest rates decrease risk taking.

In our formal analysis, we have assumed that payoffs and signals are (Log)normally distributed. While this may seem restrictive, it is quite clear that the basic intuition underlying our results carries over to all information systems and risk measures that imply single-crossing of conditional expectation functions.
A Debt Financing

Many complex financial instruments, such as mortgage-backed securities and collateralized debt obligations, are either debt instruments or some sort of hybrid between debt and equity. Here, we show that the results in the main text carry over to full or partial debt financing.

Suppose that a risk-neutral bank is asked to (co-)finance investment projects through a standard debt contract. Which projects will be financed? First, consider the case where the bank is asked to finance projects in their entirety. In that case, at the margin, debt is effectively the same as equity. To see this, consider a marginal project with \( \tilde{y}_\kappa = y^R_\kappa \), \( \kappa \in \{P,T,H,L\} \). The bank will only be willing to fully finance such a project, if the investor, who has no skin in the game, is willing to accept such a high interest rate, \( \rho \), that the bank becomes the residual claimant and, hence, the de facto equity holder. (For unbounded payoff distributions such as the Lognormal distribution, \( \rho \to \infty \). For bounded payoff distributions, the gross lending rate charged by the bank for marginal projects is equal to the upper bound of the payoff distribution.) Therefore, under full debt financing, a marginal project from the perspective of an equity investor is also marginal from the perspective of a debt investor, and vice versa. This means that exactly the same projects are undertaken regardless of whether they are financed through equity or debt contracts.

What happens if financing is partly in debt and partly in equity? Suppose the bank finances a fraction \( \delta \), \( \delta \in [0,1] \), in the form of debt, while the investor puts up the remaining \( 1 - \delta \) in the form of equity. In a competitive lending market, the lending rate charged by the bank is determined by a zero-profit condition: it charges a gross lending rate \( \rho \) such that the expected total repayment to the bank is equal to \( \delta R \)—provided, of course, that such a \( \rho < \infty \) exists. (If such a \( \rho \) does not exist, then the bank’s expected payoff is always less than \( \delta R \) in expectation and, hence, it is not willing to extend the loan.) For the equity investor to be willing to accept this lending rate, the residual payment he expects to receive, \( E[\tilde{q}_\kappa | y_\kappa] - \delta R \), must be at least as large as his gross funding cost, i.e.,

\[
E[\tilde{q}_\kappa | y_\kappa] - \delta R \geq (1 - \delta) R
\]

This is equivalent to

\[
E[\tilde{q}_\kappa | y_\kappa] \geq R
\]

which is identical to the investor’s decision rule under equity financing. Hence, also for mixed debt-equity financing, the selection of projects is the same as under full equity financing. Note that the argument holds for all types of projects, i.e., opaque, transparent, fundamentally safe, and fundamentally risky. We summarize this observation in the next proposition.

Proposition 13 The financing structure has no influence on which projects are undertaken. Hence, all propositions derived for equity financing carry over to the case of (full or partial) debt financing.

Finally, we note that \( \rho \) is decreasing in the normalized signal \( z \) and increasing in leverage \( \delta \). Indeed, both results are intuitive. They follow from the fact that an increase in \( z \) reduces the loan’s non-repayment risk, while an increase in \( \delta \) raises it. To save space, we omit formal proofs of these results.
B Proofs

Proof of Proposition 1. The probability $p^R_\kappa$ that a random project of type $\kappa$ is financed is equal to

$$p^R_\kappa = \Pr \left( y_\kappa \geq \ln R + \frac{\sigma^2_1}{\sigma^2_0} (\ln R - \nu) \right) = 1 - \Phi \left( \frac{\ln R + \frac{\sigma^2_1}{\sigma^2_0} (\ln R - \nu) - \mu}{\sqrt{\sigma^2_0 + \sigma^2_1}} \right) = 1 - \Phi \left( \frac{1}{i_\kappa} (\ln R - \nu) + \frac{1}{2} i_\kappa \right)$$

Therefore, $p^R_P > p^R_T$ iff

$$\ln R < \nu + \frac{1}{2} i_P i_T = R^*$$

This proves the equivalence between statements 1 and 2.

The equivalence between statements 2 and 3 is trivial.

Finally, to prove the equivalence between statements 3 and 4, let $s_P$ denote the share of opaque projects in investors’ portfolios and note that the average opacity of financed projects, $\hat{\sigma}^2_1$, is

$$\hat{\sigma}^2_1 = s_P \sigma^2_1 p + (1 - s_P) \sigma^2_1 P$$

$$= \frac{\alpha p^R_P}{\alpha p^R_P + (1 - \alpha) p^R_T} \sigma^2_1 p + \frac{(1 - \alpha) p^R_T}{\alpha p^R_P + (1 - \alpha) p^R_T} \sigma^2_1 T$$

$$= \frac{\alpha p^R_P / p^R_T}{\alpha p^R_P / p^R_T + (1 - \alpha)} \sigma^2_1 p + \frac{(1 - \alpha)}{\alpha p^R_P / p^R_T + (1 - \alpha)} \sigma^2_1 T.$$ 

Hence, $\hat{\sigma}^2_1 > \hat{\sigma}^2_1 = \alpha \sigma^2_1 p + (1 - \alpha) \sigma^2_1 T$ iff $p^R_P / p^R_T > 1$. The comparison between $\hat{\sigma}^2$ and $\hat{\sigma}^2$ proceeds analogously. □

Proof of Proposition 2. The probability $p^R_\kappa (q)$ that a project of type $\kappa$ with payoff $q$ will be financed is equal to

$$p^R_\kappa (q) = \Pr \left( y_\kappa \geq y_\kappa^R \mid q \right) = \Pr \left( y_\kappa \geq \ln R + \frac{\sigma^2_1}{\sigma^2_0} (\ln R - \nu) \mid q \right) = 1 - \Phi \left( \frac{\ln R - \ln q + \frac{\sigma^2_1}{\sigma^2_0} (\ln R - \nu)}{\sigma_1,\kappa} \right)$$

Therefore, $p^R_P (q) > p^R_T (q)$ iff

$$\frac{\ln R - \ln q + \frac{\sigma^2_1}{\sigma^2_0} (\ln R - \nu)}{\sigma_1, p} < \frac{\ln R - \ln q + \frac{\sigma^2_1}{\sigma^2_0} (\ln R - \nu)}{\sigma_1, T}$$
which is equivalent to
\[ \ln q < \ln R + \frac{\sigma_{1,P} \sigma_{1,T}}{\sigma_{0}^2} (\nu - \ln R) \]

Thus, if \( \ln R \geq \nu \), then \( p_{0}^{R}(q) > p_{p}^{R}(q) \) implies that \( q < R \). Hence, the project is unprofitable. If, however, \( \ln R < \nu \), then all projects with \( \ln q \in \left( \ln R, \ln R + \frac{\sigma_{1,P} \sigma_{1,T}}{\sigma_{0}^2} (\nu - \ln R) \right) \) are strictly profitable and satisfy the above inequality, such that they have a greater chance of financing if they are opaque than if they are transparent. ■

**Proof of Proposition 3.** We prove the proposition by calculating \( \frac{d}{dR} \left( \frac{p_{0}^{R}}{p_{p}^{R}} \right) \) and showing that it is positive for \( \ln R \geq \nu \).

Let \( x_{T} = \frac{1}{i_{T}} (\ln R - \nu) + \frac{1}{2} i_{T} \) and \( x_{p} = \frac{1}{i_{p}} (\ln R - \nu) + \frac{1}{2} i_{p} \), and denote by \( \Phi \) (respectively, \( \phi \)) the CDF (PDF) of the standard Normal distribution. Then,

\[
\frac{d}{dR} \left( \frac{p_{0}^{R}}{p_{p}^{R}} \right) = \frac{1}{R} \frac{\frac{1}{i_{p}} (1 - \Phi (x_{T})) \phi (x_{p}) - \frac{1}{i_{T}} (1 - \Phi (x_{p})) \phi (x_{T})}{(1 - \Phi (x_{p}))^2}
\]

This expression takes on the same sign as
\[
\frac{i_{T} l (x_{p})}{i_{p} l (x_{T})} - 1
\]

where \( l \) is the hazard rate of the standard Normal distribution.\(^9\)

If \( \ln R \) is sufficiently large such that \( x_{p} \geq x_{T} \) (which is equivalent to \( R \geq R^{*} \)), then the expression in (1) is clearly strictly positive and we are done. Hence, in the remainder, we assume that \( x_{p} < x_{T} \).

If \( \ln R = \nu \), the expression in (1) simplifies to
\[
\frac{i_{T} l \left( \frac{1}{2} i_{p} \right)}{i_{p} l \left( \frac{1}{2} i_{T} \right)} - 1
\]

which is positive, because \( \frac{l(v)}{l(w)} > \frac{v}{w} \) for all \( 0 < v < w \).

To establish the proposition, it now suffices to show that \( \frac{d l (x_{p}) / l (x_{T})}{d \ln R} > 0 \) for \( \nu \leq \ln R \leq \ln R^{*} \). We have,
\[
l^{2} \left( x_{T} \right) \frac{d l (x_{p}) / l (x_{T})}{d \ln R} = \frac{l (x_{T}) l' (x_{p})}{i_{p}} - \frac{l (x_{p}) l' (x_{T})}{i_{T}}
\]

Using \( l' (v) = l (v) (l (v) - v) \), the right-hand side of the last equality is equal to
\[
\frac{l (x_{T}) l (x_{p})}{i_{p}} \left( \frac{l (x_{p}) - x_{p}}{i_{p}} - \frac{l (x_{T}) - x_{T}}{i_{T}} \right)
\]

\[
> l (x_{T}) l (x_{p}) \left( \frac{l (x_{p}) - x_{p}}{i_{T}} - \frac{l (x_{T}) - x_{T}}{i_{T}} \right) > 0
\]

\(^9\)Recall that the hazard rate of the standard Normal distribution, \( l (v) \), has the following properties:

1. \( l (v) \) is strictly increasing and satisfies the differential equation \( l' (v) = l (v) (l (v) - v) \);
2. \( l (v) - v \) is strictly positive and strictly decreasing;
3. \( l (v)/v \) is strictly decreasing for \( v > 0 \).
where the last inequality follows from \( x_P < x_T \) and the fact that \( l (v) - v \) is strictly decreasing in \( v \).

**Proof of Proposition 4.** The expected payoff of a financed project of type \( \kappa \) is \( E \left[ e^{\tilde{\pi}_\kappa} \mid \bar{y}_\kappa \geq y^R_\kappa \right] \). Now,

\[
E \left[ e^{\tilde{\pi}_\kappa} \mid \bar{y}_\kappa \geq y^R_\kappa \right] = \int_{y^R_\kappa}^{\infty} E \left[ e^{\tilde{\pi}_\kappa} \mid \bar{y}_\kappa = y^R_\kappa \right] \frac{f(y)}{1 - F(y^R_\kappa)} dy \\
= \frac{1}{\sigma_{\kappa}^2} E \left[ e^{\tilde{\pi}_\kappa} \right] \left( \frac{1}{\sigma_{\kappa}^2} \right) \bar{y}_\kappa \mid \bar{y}_\kappa \geq y^R_\kappa
\]

Recall that if \( \ln X \sim N(m, s^2) \), then \( E [X \mid X \geq d] = e^{m+\frac{1}{2}s^2} \frac{1 - \Phi \left( \frac{\ln d - m - s^2}{s} \right)}{1 - \Phi \left( \frac{\ln d - m}{s} \right)} \). This implies that

\[
E \left[ e^{\tilde{\pi}_\kappa} \mid \bar{y}_\kappa \geq y^R_\kappa \right] = e^{\nu} \frac{1 - \Phi \left( \frac{y^R_\kappa - \nu - \frac{1}{2} \sigma_\kappa^2}{\sqrt{\sigma_\kappa^2 + \sigma_{\kappa}^2 R}} \right)}{1 - \Phi \left( \frac{y^R_\kappa - \nu + \frac{1}{2} \sigma_\kappa^2}{\sqrt{\sigma_\kappa^2 + \sigma_{\kappa}^2 R}} \right)}
\]

Therefore, \( E \left[ e^{\tilde{\pi}_T} \mid \bar{y}_T \geq y^R_T \right] > E \left[ e^{\tilde{\pi}_P} \mid \bar{y}_P \geq y^R_P \right] \) iff

\[
\frac{1 - \Phi \left( \frac{1}{1T} (\ln R - \nu) - \frac{1}{2} i_T \right)}{1 - \Phi \left( \frac{1}{1T} (\ln R - \nu) + \frac{1}{2} i_T \right)} > \frac{1 - \Phi \left( \frac{1}{1P} (\ln R - \nu) - \frac{1}{2} i_P \right)}{1 - \Phi \left( \frac{1}{1P} (\ln R - \nu) + \frac{1}{2} i_P \right)}
\]

Subbing \( y^R_\kappa = \ln R + \frac{\sigma_{\kappa}^2}{\sigma_0^2} (\ln R - \nu) \) yields

\[
\frac{1 - \Phi \left( \frac{1}{1T} (\ln R - \nu) - \frac{1}{2} i_T \right)}{1 - \Phi \left( \frac{1}{1T} (\ln R - \nu) + \frac{1}{2} i_T \right)} > \frac{1 - \Phi \left( \frac{1}{1P} (\ln R - \nu) - \frac{1}{2} i_P \right)}{1 - \Phi \left( \frac{1}{1P} (\ln R - \nu) + \frac{1}{2} i_P \right)}
\]

And since \( i_T > i_P \), this inequality follows from Lemma 2 below.

**Lemma 2** The function

\[
\theta (i) = \frac{1 - \Phi \left( \frac{1}{i} (\ln R - \nu) - \frac{1}{2} i \right)}{1 - \Phi \left( \frac{1}{i} (\ln R - \nu) + \frac{1}{2} i \right)}
\]

is strictly increasing in \( i \) for \( i > 0 \).

**Proof.** Let \( x = \frac{1}{i} (\ln R - \nu) + \frac{1}{2} i \) and \( y = \frac{1}{i} (\ln R - \nu) - \frac{1}{2} i \), and note that \( x = y + i \). Differentiating \( \theta (i) \), we get

\[
\theta' (i) (1 - \Phi (x))^2 = \phi (y) \left( \frac{1}{i^2} (\ln R - \nu) + \frac{1}{2} \right) (1 - \Phi (x)) \]

\[+ \phi (x) \left( \frac{1}{i^2} (\ln R - \nu) - \frac{1}{2} \right) (1 - \Phi (y))
\]
which has the same sign as

\[
\begin{align*}
l(y) \left( \frac{1}{y^2} \ln(R - \nu) + \frac{1}{2} \right) - l(x) \left( \frac{1}{x^2} \ln(R - \nu) - \frac{1}{2} \right) \\
= (l(y) - l(x)) \frac{1}{y^2} \ln(R - \nu) + \frac{1}{2} (l(y) + l(x))
\end{align*}
\] (2)

where \( l(\cdot) \) denotes the hazard rate of the standard Normal distribution.

Because \( x > y \), this expression is clearly positive for \( \ln R \leq \nu \). Hence, it remains to consider the case \( \ln R > \nu \).

The expression in (2) can be written as

\[
- \frac{l(y + i) - l(y)}{i} \frac{1}{x^2} \ln(R - \nu) + \frac{1}{2} (l(y) + l(y + i))
\]

\[
> - \frac{1}{i} \ln(R - \nu) + \frac{1}{i} \ln(R - \nu) = 0
\]

where the inequality follows from \( \ln R > \nu \) and the fact that \( l'(v) < 1 \) and \( l(v) > v \), for all \( v \in \mathbb{R} \).

**Proof of Proposition 6.**

The proof is analogous to that of Proposition 1.

**Proof of Proposition 7.**

The probability \( p^R_\lambda(q) \) that a project of type \( \lambda \) with payoff \( q \) will be financed is equal to

\[
p^R_\lambda(q) = \Pr(y_\lambda \geq y^R_\lambda \mid q) = \Pr \left( y_\lambda \geq \ln R + \frac{\sigma^2_1}{\sigma^2_{0,\lambda}} (\ln R - \nu) \mid q \right)
\]

\[
= 1 - \Phi \left( \frac{\ln R - \ln q + \frac{\sigma^2_1}{\sigma^2_{0,\lambda}} (\ln R - \nu)}{\sigma_1} \right)
\]

Therefore, \( p^R_H(q) > p^R_L(q) \) iff

\[
\frac{\ln R - \ln q + \frac{\sigma^2_1}{\sigma^2_{0,\lambda}} (\ln R - \nu)}{\sigma_1} < \frac{\ln R - \ln q + \frac{\sigma^2_1}{\sigma^2_{0,\lambda}} (\ln R - \nu)}{\sigma_1}
\]

which is equivalent to

\[ \ln R > \nu \]

**Proof of Proposition 8.** The proof is analogous to that of Proposition 3.

**Proof of Proposition 9.** The proof is analogous to that of Proposition 4.
Proof of Lemma 1. The result immediately follows from the fact that

\[
\frac{dp_R}{di} = \frac{1}{i^2} \left( \ln R - \left( \nu + \frac{1}{2}i^2 \right) \right) \phi \left( \frac{1}{i} \left( \ln R - \nu \right) + \frac{1}{2}i \right)
\]

\]

Proof of Proposition 12. From Proposition 11 we know that, for \(\ln R < \nu + \frac{1}{2}i^2\), \(\frac{dp_R}{di} < 0\) for all \((\sigma_0^2, \sigma_1^2) \in [\sigma_0^2, \sigma_0^2] \times [\sigma_1^2, \sigma_1^2]\). Hence, for these values of \(R\), \(p^R(\sigma_0^2, \sigma_1^2)\) is strictly increasing in \(\sigma_1^2\) and strictly decreasing in \(\sigma_0^2\). Independence between \(\tilde{\sigma}_0^2\) and \(\tilde{\sigma}_1^2\) then implies that \(\tilde{\sigma}_1^2\) and \(\tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)\) are strictly positively correlated. Thus,

\[
\hat{\sigma}_1^2 = \int_{\sigma_0^2}^{\sigma_1^2} \int_{\sigma_0^2}^{\sigma_1^2} \sigma_1^2 h(\sigma_0^2, \sigma_1^2) \, d\sigma_1^2 \, d\sigma_0^2
\]

\[
= \int_{\sigma_0^2}^{\sigma_1^2} \int_{\sigma_0^2}^{\sigma_1^2} \int_{\sigma_0^2}^{\sigma_1^2} \int_{\sigma_0^2}^{\sigma_1^2} \frac{p^R(\sigma_0^2, \sigma_1^2) g(\sigma_0^2, \sigma_1^2)}{\sigma_1^2} \, d\sigma_1^2 \, d\sigma_0^2
\]

\[
= \frac{E[\tilde{\sigma}_1^2 \tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)]}{E[\tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)]} = \frac{\hat{\sigma}_1^2 E[\tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)] + Cov(\tilde{\sigma}_1^2, \tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2))}{E[\tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)]}
\]

\[
= \hat{\sigma}_1^2 + \frac{Cov(\tilde{\sigma}_1^2, \tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2))}{E[\tilde{p}^R(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)]} > \hat{\sigma}_1^2
\]

A similar argument establishes that \(\hat{\sigma}_0^2 < \hat{\sigma}_1^2\).

The proof that for \(\ln R > \nu + \frac{1}{2}i^2\), \(\hat{\sigma}_1^2 < \hat{\sigma}_1^2\) and \(\hat{\sigma}_0^2 > \hat{\sigma}_0^2\) is analogous. \(\blacksquare\)
References


