Sovereign default and public debt sustainability.*

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Abstract

We address sovereign default in a stochastic macroeconomic model with infinite horizon and the presence of a debt recovery rule. Sovereign default is defined as a market event which can be observed when the fiscal policy is constrained by a tax rate ceiling. It occurs when the fiscal authority does not find on the market the funds necessary to reimburse its previous debt net of the primary surplus of the period. We analyze the pricing of public debt in relation with the debt recovery rule and prove the existence of a threshold default. Except in a specific case, this threshold is lower than the traditional solvency ratio. Using the dynamic implications of the model, we clarify the notion of sustainability and we disentangle it by defining a sustainability threshold and an un-sustainability threshold: A public debt is said to be "$\underline{\varphi}-$sustainable" at date t when its trajectory does not reach the default threshold at any future date, assuming that there is no realization of the gross rate of growth lower than $\underline{\varphi} \leq 1$. A public debt is said to be "$\overline{\varphi}-$unsustainable" at date t when its trajectory reaches the default threshold at some finite date, assuming that there is no realization of the gross rate of growth higher than $\overline{\varphi} \geq 1$. When a public debt is neither "$\overline{\varphi}-$unsustainable", nor "$\underline{\varphi}-$sustainable", it is in a zone of financial fragility. When a sovereign default occurs, a too high recovery ratio is not able to insure the sustainability of the post-default debt.

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1 Introduction

In this paper we offer a new analysis of the dynamics of public debt which takes into consideration the possibility of sovereign default and the "rescheduling" of the debt. This leads us to define the default threshold in accordance to the market pricing of debt and the stochastic nature of the economy: Default occurs when lenders are unwilling to pursue lending to the state, that is, consider that the government is unable to fulfill its contractual debt obligation and service it in the future: the debt burden has reached its upper level; beyond it, the state is forced to default. The sustainability of public debt is embedded in a truly stochastic environment and we distinguish the sustainability and the unsustainability thresholds, each related to the sequence of shocks. Finally we highlight the role of the debt recovery ratio on the whole dynamics of public debt, both before and after default and we prove that for a given sequence of shocks, this ratio must be low enough for the public debt to be sustainable, that is, expected to converge toward a stable steady-state.

Macroeconomists are used to reflect on the sustainability of public debt but not to address squarely issues related sovereign default. It is common to rule out default by imposing the sustainability of public debt. But in an uncertain world this requirement cannot hold.[1] *Sovereign defaults happen and the evidence is overwhelming.* [2] *Some countries have experienced the recurrence of default ("serial defaults") as documented by Reinhard and Rogoff (2008). Puzzlingly, countries do not default under similar circumstances: The debt-GDP ratios at which defaults occur vary. In other words, some countries are easily subject to default, or "default intolerant", whereas others seem in the same circumstances not to be subject to default attacks (see Reinhart, Rogoff and Savastano, 2003).*

Most of the theoretical literature on default focuses on solving the puzzle of the existence of sovereign debt contracts between fully rational agents when there is no or limited enforcement capacity, following Eaton and Gersovitz (1981) and has followed a microeconomic approach to discuss this problem.[3] *Needless to say, it is central to reflect on the bargaining between lenders and sovereign borrowers, public debt contracts and the foundations of debt recovery rules. However a macroeconomic analysis is also needed because default is clearly a macroeconomic phenomenon: a sovereign is a major macroeconomic agent, macroeconomic policies are affected both by the probability of future defaults (either because they are modified or because the macroeconomic environment in which they operate is altered) and by the decision to default which in general goes along to a new policy set. Hence it poses a twin challenge to macroeconomists: what leads to default and is default a single exceptional event? First, we need to know which circumstances lead to sovereign default. Is it bad luck, inadequate macroeconomic policies or improper financial tools and circumstances? Second, we need to know what*

[1] *See Bohn (2008).*

[2] *See Reinhard and Rogoff (2008).*

[3] *See Panizza, Sturzenegger and Zettelmeyer (2009) for a recent survey of this literature. See also Jonathan Eaton and Raquel Fernandez (1995) and Kenneth M. Kletzer (1994).*

to do after default: whether the sovereign is able to restore its credit and borrow again without the prospect of future default, or whether the dynamics of public debt after default leads unavoidably to a future default. Once a default has occurred, a given debt recovery rule is applied[4]*, which in effect specifies an "haircut" ratio to be applied to the contractual obligations of the borrower.*

The aim of this article is to answer these questions in a dynamic stochastic macroeconomic model, with indefinitely lived agents forming rational expectations which we are able to solve analytically. So we not address the occurrence of a unique default at a given period, but the logic of default, leading to the possible recurrence of defaulting events over time. We stress the role of the debt recovery rule in the answers to both questions. This comes from the fact that it affects the risk premium to be included in the interest rate applied to public debt. Hence it plays a role not just after default has occurred and the debt recovery rule is applied but before, because the prospect of default and the anticipation of an haircut on the contractual debt determines the expected loss, to be countered by a higher risk premium. The weight of the service of debt in the accumulation of debt is well known and understood. Here we are able to properly analyze it in a macro model.

Our model hinges on the existence of a maximum tax rate which cannot be trespassed. This creates a kink in the dynamic equation of expected emitted debt which allows us to define two steady states, a low debt-to-GDP ratio steady state which is stable, the other one characterized by a high debt-to-GDP ratio and unstable. This duality is central in the dynamics of public debt. It explains why default can occur, but infrequently: affected by external shocks, public debt may decrease and be driven toward the stable steady state or increase and be led to eventual default. But actually the high debt-to-GDP ratio steady state does not define market-driven default. Within our model, we are able to characterize the upper limit for the debt-to-GDP ratio, which we call the "equilibrium default threshold". The default threshold corresponds to the maximum stationary amount of debt which can be absorbed by the financial market. It depends on the risk premium applied to public debt and thus on the debt recovery rule which impacts (over time) on the risk borne by the lenders. By definition, it is impossible to trespass this threshold without defaulting. This happens because of three factors: a fiscal policy which has reached the tax rate limit, adverse shocks which pushed up the deficit, and the debt recovery ratio.

The question now is whether the dynamics of public debt is such that it heads toward this threshold or not. We first study the impact of the stochastic process and shocks on the sustainability of public debt. To answer this problem we offer a new approach to public debt sustainability. It has the characteristics of not ruling out a priori the possibility of default but rather corresponds to the avoidance of market-driven sovereign defaults. This approach leads us to prove the existence of two critical levels of debt:

1. *the "$\overline{\varphi}-$unsustainable level of debt": Above this level, assuming that the future*

[4] *This debt recovery rule is usually the result of a lengthy bargaining process. Here we do not study this process and rely on an ad hoc rule.*

growth rates are never above $\overline{\varphi}$, public debt is unsustainable and default will eventually occur, whatever the highness of $\overline{\varphi}$.

2. *the "$\underline{\varphi}-$sustainable level of debt": Below this level, assuming that the future growth rate is never lower than $\underline{\varphi}$, public debt is sustainable as it converges to toward the stable steady-state, whatever the lowness of $\underline{\varphi}$.*

The $\overline{\varphi}-$unsustainability of public debt is due to the snowball effect of risk premium: above this critical value, given the debt recovery rule, the risk premium necessary to clear the debt market is too high, the public debt balloons and eventually triggers a new default. Similarly the "$\underline{\varphi}-$sustainable level of debt" is sufficiently low, so that servicing the debt is low enough, and despite the low fiscal inflows, the public debt decreases gradually and converges to a stable steady-state.

Given the impact of the debt recovery rule, it is important to understand its impact on the dynamics of public debt. Empirically, some countries are unable to avoid defaulting repeatedly. Often the defaulting process is characterized by "too little, too late": it happens too late, the debt reduction is too little. Our model vindicates the last stylized fact as we prove that for a sufficiently large debt reduction, the post-default debt will be below the $\underline{\varphi}-$sustainable level and thus be $\underline{\varphi}-$sustainable.

In brief, our analysis sheds light on the various factors leading to default and uncovers important trade-offs related to the debt recovery rule. This leads to show how the sustainability of public debt is linked to the policy structure, the occurrence of shocks and the debt recovery rule applied after default and how to disentangle the impacts of these factors on the dynamics of public debt.

As we focus on the inclusion of the various channels linking sovereign default and the dynamics of public debt we abstain from looking at the decision to default or to the bargaining process leading to a reduction in the financial obligations of the sovereign. In other words, there is nothing strategic in our setting. The government is passive in the following sense: public expenditures are proportional to output and a fiscal rule generates the tax inflows. Deficits are covered by emitting debt on the financial market. If no potential lender is willing to buy the emitted debt, there is default. If this happens, a debt reduction rule is automatically and immediately applied. Our perspective is thus very different from most analyses of sovereign default which focus on the intertwined decisions to default and bargain on a financial contract under which sovereign debt is issued.

The paper is organized as follows. We discuss related literature in the following section. Section 3 presents the macroeconomic framework. The existence of a maximum amount of debt beyond which there is default, what we define as the "default threshold", and its properties are discussed in section 4. The dynamics of public debt when default is not ruled out is addressed in Section 5. Section 6 concludes.

Related literature.

Uribe (2006) is the first to offer a (fiscal) theory of sovereign default in a monetary macroeconomic model, similar to the one analyzed by Woodford (1995). Uribe focuses on the possibility of default in the case when both fiscal and monetary policies are active (in the sense of Leeper) or dominant (in the sense of Sargent and Wallace) and develops a "fiscal theory of default".[5] *He shows that, when explicitly introducing the possibility of actual default, default occurs as an adjustment variable. In the presence of shocks, default is observed at each period, allowing inflation to remain at its target value and public debt not to overrun its maximum sustainable value.*[6] *However in the case of positive shocks, a "negative" default occurs in this model. Lenders receive more than the contractual debt to be reimbursed. This feature is irrealistic and in addition, it nullifies any risk premium since positive default is as probable as negative default (assuming normal shocks). Interestingly Uribe shows that if default is delayed (in the case of negative shocks), inflation may temporarily increase until default, the magnitude of which is then much more important than in the case of an immediate default.*

Uribe's analysis relies on the importance of the role of the transversality condition in the economy he considers. Given the uniqueness of the steady state equilibrium, a decreasing debt converging to this steady state level is as likely as an unsustainable increasing debt. Fiscal policy is obliged to be passive only locally, because of the existence of a high level of debt and a fiscal limit. Arellano (2008) studies default in a stochastic general equilibrium model with endogenous default risk and shows that default is more likely in recessions. Cuadra et al. (2010) study theoretically and quantitatively the links between fiscal variables, sovereign interest rate spreads and default risk by means of a dynamic stochastic small open economy model with incomplete markets and endogenous fiscal policy with the aim of capturing stylized facts characteristics of emerging market economies, such as the procyclicity of fiscal policy. However in their model a defaulting country resorts to temporary autarky and the impact of the renegotiation process over the dynamics of the macroeconomy is not addressed. Mendoza and Yue (2012) set up a DSGE model with default which provides an explanation for the negative relationship between sovereign spreads and GDP growth but take as given the threshold levels linked to default.

More recently, Juessen, Linnemann and Schabert (2010) address the impact of fiscal policy and limits on default, stressing the role of the Laffer curve and showing the relationship between output variability and risk premia. However they assume that after default, a country switches temporarily to autarky and not that debt is rescheduled. Bi (2012) explores the link between the fiscal capacity of a government, the composition of public spending and sovereign default but resorts to simulation techniques to study the non-linear relationship between the risk premium and the debt level. Davig, Leeper and Walker (2011) discuss the needed adjustments of monetary and fiscal policies to avoid default.

[5] *Blanchard (2004) and Loyo (1999) elaborate on similar grounds a "fiscal theory of inflation"*
[6] *This value corresponds to* ω^{sup} *in our model, which is not the default threshold. See below.*

Yue (2010) explicitly addresses the issue of debt renegotiation after default, by means of a Nash bargaining procedure and shows the interaction between sovereign default and ex post debt renegotiation. But she does not relate default to macroeconomic policies and resorts to simulations to analyze the dynamics of her small open economy.[7]

To the best of our knowledge, the issues of the impact of the debt recovery rule on the whole dynamics of the macroeconomy and the occurrence of future defaults have not been addressed in the literature.

2 The model.

We consider a closed economy with flexible prices and no capital. Money plays no role and prices are expressed in terms of the consumption good. Financial markets are complete and public bonds are non-contingent and potentially subject to a risk of default.

The absence of explicit nominal considerations can be justified by assuming that the monetary authority is independent and is able and willing to set the net inflation rate at zero. This comes at a cost: we do not investigate the impact of inflation on the unsustainability of debt, via the inflation premium inserted into the interest rate, nor the possibility to use seignorage as a way to supplement fiscal receipts and contribute to the financing of debt. We choose to neglect the monetary dimension of the problem for tractability and in order to focus on the prime responsibility of fiscal policy on the dynamics of public debt and default.

The completeness of financial markets is ensured by the presence of Arrow-Debreu contingent assets. In addition to these assets, there exists a public bond through which the sovereign obtains funds from the financial markets. This coexistence allows lenders to arbitrage and be potentially protected from the adverse consequences of default. This choice has the nice property to allow us to consistently and systematically study the intertemporal dynamics of debt at any period, before or after a default.

We abstain from any analysis of the negotiations between lenders and the sovereign borrower in case of default. We assume that a debt recovery rule is established, is valid in any period and is agreed by all parties. We take it as given and do not analyze its merits nor the process through which it was established. In other words, we do not include in our analysis any strategic consideration. As said above, many analyses have been devoted to this issue and generated important and illuminating considerations on this matter. But our sole focus is on the dynamics of public debt in the presence of default. This explains why we reason on a given debt recovery rule, defining the post-default initial level of debt to be serviced by the sovereign. It is a simple rule encapsulating the notion of "haircut". Other rules could be analyzed using our methodology.

[7] *See also Guimaraes (2011) who reverts to the assumption of full bargaining power to the creditors.*

2.1 Private sector.

There is a representative agent whose preferences are represented by the following utility function:

$$U_0 = E_0 \sum_{t=0}^{+\infty} \beta^t \left[u\left(C_t\right) - v\left(L_t\right)\right], \tag{1}$$

with: $u\left(C_t\right) = \ln C_t$ and $v\left(L_t\right) = \eta^{-1} L_t^{1+1/\sigma} / \left(1 + 1/\sigma\right)$, where C_t is consumption, L_t represents hours worked, and σ the Frisch elasticity.

In each period the agent receives profits Π_t and labor income $W_t L_t$, where W_t denotes the real wage rate. Income, including profits but excluding other financial returns for sake of simplicity, is taxed at a proportional rate τ_t. The consumer can save by means of a portfolio of Arrow-Debreu state-contingent assets and one-period maturity Treasury bonds. The amount of new issued government bonds she chooses to buy in t is noted B_t and their unit price is q_t. The amount of redeemed debt is denoted by $h_t B_{t-1}$ where h_t denotes the fraction of debt actually reimbursed. It is less than 1 in the case of default. Denoting by $Q_{t,t+1}$ the price of a contingent asset which generates a real return of 1 in a given state of nature (and 0 in the others) divided by the probability (or density function) of such state,[8] and by D_{t+1} the quantity of this contingent asset,[9] the individual budget constraint at t writes:

$$C_t + q_t B_t + E_t\left(Q_{t,t+1} D_{t+1}\right) \leq \left(1 - \tau_t\right)\left(W_t L_t + \Pi_t\right) + h_t B_{t-1} + D_t. \tag{2}$$

The agent must also meet her intertemporal constraint on wealth:

$$h_{t+1} B_t + D_{t+1} \geq -E_{t+1} \sum_{s=t+1}^{\infty} Q_{t+1,s}\left(1 - \tau_s\right)\left(W_s L_t + \Pi_s\right) \qquad \forall t+1, \tag{3}$$

with $Q_{t+1,s} \equiv Q_{t+1,t+2} Q_{t+2,t+3} \cdots Q_{s-1,s}$ and $Q_{t+1,t+1} = 1$. This condition must hold for each possible state that may occur at date $t+1$.

Maximizing (1) under contraints (2) and (3), the following optimality conditions obtain at every period t:

$$Q_{t,t+1} = \beta \frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)} = \beta \frac{C_t}{C_{t+1}}, \tag{4}$$

$$q_t = E_t Q_{t,t+1} h_{t+1}, \tag{5}$$

$$\left(1 - \tau_t\right) W_t = \frac{v'\left(L_t\right)}{u'\left(C_t\right)} = \frac{L_t^{1/\sigma}}{\eta} C_t \tag{6}$$

[8] Which will be equal to the stochastic discount factor.

[9] For the sake of simplicity, we do not use notation for the states of Nature that may occur at each date. Remember that there are as many different values for D_{t+1} and $Q_{t,t+1}$ as possible states of Nature in $t+1$. The contingent asset is indexed by $t+1$ since its return will depend on the state of nature realized in $t+1$. To the contrary the public bond emitted in t is indexed by t as it is not state-contingent in $t+1$.

and the transversality condition is given by:

$$\lim_{T \to \infty} E_t Q_{t,T} \left[h_T B_{T-1} + D_T \right] = 0. \tag{7}$$

(4) is the state contingent Euler equation for consumption. This condition must hold for each possible state that may occur at date $t+1$, given the state that has occurred at date t. (5) equates the price of the risky government bond to the expected discounted return of the reimbursed debt next period. The RHS of this equation can be interpreted as the value of a specific portfolio composed of contingent assets, each one bought in quantity h_{t+1}. Hence (5) is the no-arbitrage condition between the risky government bond and this particular portfolio. Finally, (6) is the intratemporal optimal condition between labor and consumption.

The good market is perfectly competitive and returns to scale are constant. The production technology is given by:

$$Y_t \le A_t N_t \tag{8}$$

where Y_t denotes production, N_t is the quantity of labor hired by the firm, and A_t is the average (and marginal) productivity of labor. It is stochastic and is the sole shock present in this economy. Profit maximization leads to standard results on returns: $W_t = A_t$, $\Pi_t = 0$ and (8) binds.

We focus on the innovation process driving production. In order to focus on the impact of a shock on the debt-to-GDP ratio, we need to generate a dynamics of output corresponding to a unit root. Specifically, we assume the following:

Assumption 1.

$$A_t = a_t A_{t-1},$$

where a_t is an i.i.d. *random variable. The cumulative distribution function of a_t is denoted by $G(a)$, its density function by $g(a)$ and we assume that:*

1. *the support of $g(a)$ is bounded on the interval $[a_{\inf}, a^{\sup}]$. In addition, $0 < a_{\inf} < 1 < a^{\sup}$ and*

$$E(a_t) = 1 \qquad \text{and} \qquad \beta E\left(\frac{1}{a_t}\right) < 1,$$

2. *$g(a) > 0$; $\lim_{a \to a^{\sup}} g(a) = \lim_{a \to a_{\inf}} g(a) = \varepsilon$ with ε arbitrarily small,*

3. *the elasticity of the density function $g(a)$ satisfies:*

$$\frac{a g'(a)}{g(a)} > -1.$$

Assumption **1.1** makes clear that the productivity follows a random walk but that the growth rate of productivity is bounded. Assumption $\beta E\left(\frac{1}{a_t}\right) < 1$ will guarantee the existence of a positive risk-free interest rate for this economy when there is no risk of default.

Assumptions **1.2** and **1.3** are regularity assumptions which allow to exclude the possibility of multiple equilibria as will be made explicit in Section 4.

2.2 Fiscal policy.

Government spends an amount $G_t = gY_t$, and collects taxes on income $\tau_t Y_t$. It balances its budget by issuing one-period maturity Treasury bonds at a price q_t. In case of default at t, it reimburses a fraction $h_t < 1$ of its debt contracted at $t-1$, B_{t-1}. The instantaneous government budget constraint writes:

$$q_t B_t = h_t B_{t-1} + (g - \tau_t) Y_t, \tag{9}$$

with $h_t \in (0,1)$.

Fiscal rule and fiscal constraint.

Following Bi (2012), Daniel and Shiamptanis (2010) and Davig, Leeper and Walker (2011), we assume that the tax rate increases with the fraction of debt to GDP, up to a limit denoted by $\hat{\tau}$. An obvious candidate for this limit corresponds to the rate generating the highest point of the Laffer curve.[10] But this limit can also be the consequence of political economy or constitutional considerations. If this is the case, it is considered here as given. When the tax rate has reached its maximum value, we refer to the situation as *fiscally constrained* and we will say that the economy is in a *constrained fiscal regime.*

We assume that the tax rate depends on $h_t B_{t-1}/Y_t$, the *actually redeemed* debt-to-output ratio,[11] as long as the upper limit $\hat{\tau}$ is not yet reached,

$$\tau_t = \min\left(\bar{\tau} + \theta \cdot \left(\frac{h_t B_{t-1}}{Y_t} - \bar{\omega}\right); \hat{\tau}\right), \tag{10}$$

and we make the following assumption:

Assumption 2. *$\theta > 1 - \beta$, $\bar{\omega} \geq 0$, and $\hat{\tau} > \bar{\tau} = g + (1-\beta)\bar{\omega}$.*

Under Assumption 2, the term $\bar{\omega}$ can then be interpreted as a target value for the actually redeemed debt-to-output ratio. From (10), we define another debt-to-output ratio $\hat{\omega}$ at which the tax rate reaches its maximum $\hat{\tau}$:

$$\tau_t = \hat{\tau} \iff \frac{h_t B_{t-1}}{Y_t} \geq \bar{\omega} + \frac{\hat{\tau} - \bar{\tau}}{\theta} \equiv \hat{\omega}. \tag{11}$$

[10] More precisely we shall see below that there exist dynamic Laffer curves in the sense that the shape of the Laffer curve depends on the state of the economy, as in Bi (2012). In contrast with Bi, and because we consider a non-stochastic fiscal policy, the maxima of these curves are obtained for a unique tax rate.

[11] The redeemed debt is possibly affected by default when $h_t < 1$.

Sovereign default and debt recovery rule.

We assume that the government does not behave strategically: it cannot decide to default discretionarily. To the contrary, default is a market event triggered by the unwillingness of savers to buy all government bonds.

Let us denote by $\Omega_t^{\max}$ the maximum debt level which can be redeemed by the Treasury without default in t. Default occurs when $B_{t-1} > \Omega_t^{\max}$. We refer to $\Omega_t^{\max}$ as the *default threshold* for period t.

As we do not focus upon the strategic relationships between lenders and the public borrower, we assume a given debt recovery rule. In case of default, a simple rule, contingent on the level of contractual debt B_{t-1} and on the default threshold $\Omega_t^{\max}$, is applied. We use the following specification for tractability reasons:

$$h_t = \mathcal{H}\left(B_{t-1}, \Omega_t^{\max}\right) \equiv \begin{cases} \mathbf{h} \cdot \Omega_t^{\max}/B_{t-1} < 1 & \text{if } \Omega_t^{\max} < B_{t-1}, \\ 1 & \text{if not,} \end{cases} \tag{12}$$

with $0 \leq \mathbf{h} \leq 1$.

According to this rule, any contractual debt level beyond the threshold $\Omega_t^{\max}$ triggers default and rescheduling. This rescheduling is such that the after-default (redeemed) debt-to-GDP ratio is a fraction of $\Omega_t^{\max}$, *i.e.*: $h_t B_{t-1} = \mathbf{h}\Omega_t^{\max}$. If we consider the limit case where the overrun is negligible ($B_{t-1} \to \Omega_t^{\max +}$), $\mathbf{h}$ can be interpreted as the maximum redemption ratio. By extension, $1 - \mathbf{h}$ is the minimal rate of default –or adjustment rate–, loosely speaking, the lowest possible "haircut".[12] In addition to its tractability this recovery rule has the property of ensuring that the government will be able to enter the bond market as its post-default initial debt is below $\Omega_t^{\max}$ and thus the economy functions again according to the set of equations characterizing its dynamics.[13] It is also consistent with the evidence that the ratio of recovered to emitted debt h_t is not unique and varies according to countries and circumstances.

2.3 Equilibrium.

At this stage, it is useful to reason on the functioning of the economy taking as given the default threshold in each period, that is the stochastic sequence $\{\Omega_t^{\max}\}$. In the following section we shall investigate this sequence based on the model itself, that is, we shall endogenize it.

One can define a competitive equilibrium for this economy, contingent to a sequence of default thresholds as follows: It is a sequence of prices $\{W_t, q_t, \{Q_{t,t+1}\}\}_{t=0}^{\infty}$, policy instruments $\{\tau_t, h_t\}$, and quantities $\{N_t, Y_t, C_t, B_t, \{D_{t+1}\}\}_{t=0}^{\infty}$ such that, for all possible sequences of exogenous shocks $\{A_t\}_{t=0}^{\infty}$ and default thresholds $\{\Omega_t^{\max}\}_{t=0}^{+\infty}$, households and firms solve their respective optimization problems, the accumulation equation of

[12] Haircuts in sovereign debt restructuring for emerging market economies over 1998 and 2005 varied from 5% in Dominican republic to 72% in Argentina (see Sturzenegger and Zettelmeyer, 2008). See also Tomz and Wright (2012).

[13] Notice that the possibility of future defaults is not ruled out.

public debt holds, the taxation and default rules hold, and all markets clear. The market clearing conditions for respectively the good market, the labor market and the contingent asset market are

$$C_t = (1-g)\,Y_t, \tag{13}$$

$$L_t = N_t, \tag{14}$$

and

$$\{D_{t+1}\} = \{0\}, \tag{15}$$

for all t.

For a given stochastic process for the exogenous shocks $\{A_t\}_{t=0}^{\infty}$ and a stochastic process for $\{\Omega_t^{\max}\}_{t=0}^{+\infty}$, the equilibrium conditions are reduced to the following set of equations:

$$Y_t = \left(\frac{\eta}{1-g}\right)^{\frac{\sigma}{1+\sigma}} (1-\tau_t)^{\frac{\sigma}{1+\sigma}} A_t, \tag{16}$$

$$q_t = \beta E_t \frac{Y_t}{Y_{t+1}} h_{t+1} \tag{17}$$

$$q_t B_t = h_t B_{t-1} + (g-\tau_t)\,Y_t, \tag{18}$$

$$h_t = \begin{cases} \mathbf{h}\cdot \Omega_t^{\max}/B_{t-1} < 1 & \text{if } B_{t-1} > \Omega_t^{\max}, \\ 1 & \text{if not,} \end{cases} \tag{19}$$

where τ_t is given by (10), and the transversality condition

$$0 = \lim_{T\to\infty} \beta^T E_t \left[\frac{h_T B_{T-1}}{Y_T}\right]. \tag{20}$$

Combining (6), (8) as an equality, (13), (14), with the conditions on returns gives (16). Combining (4), (5), and (13) generates the no-arbitrage condition (17). (18) is the government budget constraint, and (19) is the default rule.

One can easily check that this economy displays a Laffer curve: the total amount of taxes collected by the government, $T_t = \tau_t Y_t$, is a non-monotone function of τ_t. In each period it is affected by the state of the economy, that is the realization of the shock A_t; however the tax rate for which it reaches its maximum is given by $\tau^{\max} = (1+\sigma)/(1+2\sigma)$ which is state-independent. $\tau^{\max}$ represents an upper value for the fiscal limit parameter $\hat{\tau}$. Notice that when $\tau_t = \tau_{t-1} = \hat{\tau}$, the gross rate of output growth is equal to $Y_t/Y_{t-1} = A_t/A_{t-1}\,(\equiv a_t)$ and follows an exogenous stochastic process.

3 Public debt dynamics: a naive approach.

Let us denote by $\omega_t \equiv h_t B_{t-1}/Y_t$, the actually redeemed debt-to-output ratio. Combining the government budget constraint (18) with (17), and divided the result by Y_t, we get:

$$\beta E_t \omega_{t+1} = \omega_t + g - \tau_t. \tag{21}$$

(21) can be read as the dynamic equation for expected actually redeemed debt-to-output ratio. Notice that the possibility of default does not appear explicitly in this equation. This is due to the combination of two elements: first, (17) means that the possibility of default is included into the pricing of public bond; second, in (21) we reason on the actually redeemed debt-to-output ratio which encompasses the possibility of default. Given the solely implicit presence of default in (21), focusing on this equation corresponds to a naive approach on the dynamics of public debt as it neglects the weight on default through the pricing of debt. Yet it uncovers an important feature of this economy.

For a given value of the tax rate, $E_t \omega_{t+1}$ is a linear function of ω_t. This is due to the logarithmic specification of the utility function in consumption and the assumption that g is constant, which makes consumption proportional to output. As we look at the expected actually redeemed debt-to-output ratio, this expression is independent of the distribution of future shocks.

Using the taxation rule (10) in equation (21) and using the definition of $\hat{\omega}$ given by (11), we get the following dynamic equation for expected actually redeemed debt-to-output ratio:

$$E_t \omega_{t+1} = \begin{cases} (1-\theta)\,\beta^{-1}\omega_t + \left(1-(1-\theta)\,\beta^{-1}\right)\bar{\omega} & \text{for } \omega_t \leq \hat{\omega}, \\ \beta^{-1}\omega_t - \beta^{-1}\left(\hat{\tau}-g\right) & \text{for } \omega_t > \hat{\omega}. \end{cases} \tag{22}$$

(22) makes clear the consequence of a maximum tax rate. It creates a kink in the dynamics of expected debt. If the actually redeemed debt-to-output ratio ω_t is sufficiently low, negative shocks on output and the resulting reduction in tax receipts can partially be offset by an increase in the tax rate as there is room to modify it. When ω_t has reached the debt-to-output ratio $\hat{\omega}$ –at which the tax rate reaches its maximum $\hat{\tau}$–, then this possibility is foregone and a negative shock on output and the ensuing deficit can only be accommodated by an increase in public debt.

The kink at $\hat{\omega}$ creates an important characteristics in the dynamics of public debt. When the actually redeemed public debt ratio ω_t is less than $\hat{\omega}$, the expected actually redeemed debt ratio is obtained from a linear equation the slope of which $((1-\theta)\,\beta^{-1})$ is less than one (from Assumption 2). When it is above it, the expected actually redeemed debt ratio is obtained from a linear equation the slope of which (β^{-1}) is more than one. Hence the kink creates two deterministic steady states, one of which is $\bar{\omega}$. The second deterministic steady state is defined by the following

$$\omega^{\sup} \equiv \frac{\hat{\tau}-g}{1-\beta}. \tag{23}$$

$\omega^{\sup}$ is equal to the sum of expected discounted primary surpluses (relative to GDP), when they are set at their maximum value; hence it defines the conventional solvency limit of public debt(-to-output ratio). In the sequel, we will refer to $\omega^{\sup}$ as the *solvency ratio* of sovereign debt. From Assumption 2, as $\bar{\omega} < \hat{\omega} < \omega^{\sup}$ the expected dynamics of the actually redeemed debt-to-output ratio may be represented by figure 1.

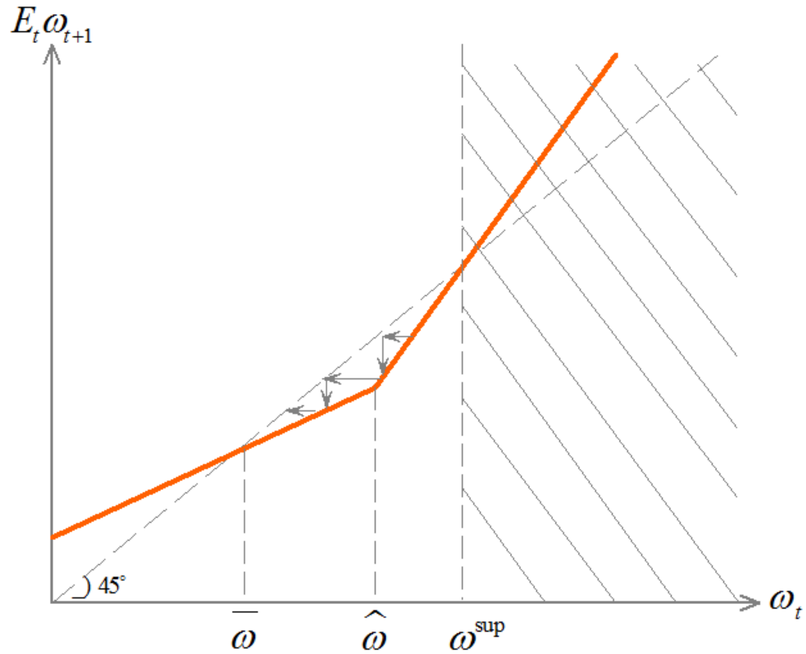

Figure 1: Deterministic Steady States

Dynamically speaking, the first deterministic steady state is stable, whereas the second one, ω^{sup}, is an unstable steady state in the following sense: If current debt ratio is less than ω^{sup}, it is expected to converge toward $\bar{\omega}$, absent of any future shock; if it were more than ω^{sup}, it is expected to grow indefinitely and violates the transversality condition.[14] Indeed, along the fiscally constrained dynamics, given by the second branch of (22), and using (23), the transversality condition (20) which can simply be written:

$$\lim_{T\to\infty} \beta^T E_t \omega_T = \omega_t - \omega^{\text{sup}} = 0,$$

is violated when $\omega_t > \omega^{\text{sup}}$.

However we need to remember that ω_t is a stochastic variable and, as such, it may "jump" in each period according to the supply shock innovation and the possibility of a sovereign default. Thus the previous analysis is incomplete and does not allow yet to get the stochastic dynamics of public debt. In addition, the dynamics of expected actually redeemed debt-to-output ratio, $E_t\omega_{t+1} \equiv E_t h_{t+1} B_t / Y_{t+1}$, does not reflect the evolution of B_t/Y_t, the level of contractual government debt emitted today relative to GDP in t. As we will see, this ratio depends from the price at which the sovereign bond will be sold and thus depends from the risk premium attached to this asset. In the next section, we will observe that the dynamics of B_t/Y_t is not necessarily stable (in a usual sense[15]) for levels of ω_t satisfying $\omega_t < \omega^{\text{sup}}$, even in relatively "good" states of nature.

[14] It is standard in many macroeconomic analyses to confound the notions of solvency and sustainability of public debt. We shall see that this confusion is misleading.

[15] That is, convergent toward $\bar{\omega}$ in absence of "bad" shocks.

4 Sovereign default and the market value of public debt.

In this section, we study the determination of the market value of public debt, taking as given the default threshold, and thus we endogenize this default threshold.

In the sequel, we restrict the analysis to configurations which fulfill the following assumption:

Assumption 3. *The economy in period t is such that:*

1. $min\left(\omega_{t-1},\omega_t\right) > \hat{\omega}$,

2. $\exists\omega_t > \hat{\omega}$ *such that:* $prob\left\{default\ in\ t+1\,|\omega_t\right\} = 0$.

Assumption 3.**1** means that the economy in period t is already in the constrained fiscal regime since at least one period, implying that $\tau_t = \tau_{t-1} = \hat{\tau}$.

Assumption 3.**2** allows us to consider the case where, despite being in the constrained fiscal regime, that is $\omega_t > \hat{\omega}$, there exist some (sufficiently low) debt-to-output ratios such that the probability of sovereign default in $t+1$ is still zero. Appendix (??) gives conditions on the parameters set under which Assumption 3.**2** holds.

Assumption 3 allows us to restrict the analysis of sovereign default to the fiscal constrained regime. We do not mean that default can only occur in such a regime. It is conceivable that the government will default when the public debt-to-output ratio is relatively low (i.e. $\omega_t < \hat{\omega}$). This may happen because of a negative shock so large that it leads to financial needs, present and future, which cannot be met by lenders. But it is an extreme case which is not the most relevant. It makes more sense to think that the fiscal authority is creeping toward default by reaching the constrained fiscal regime and thus being more exposed to the adverse consequences of a negative shock.

Let us denote by $b_t = B_t/Y_t$, the level of contractual government debt emitted today relative to GDP at t and by $\omega_t^{\max}$ the default threshold for period t as a percentage of GDP, that is: $\omega_t^{\max} \equiv \Omega_t^{\max}/Y_t$. Given the definition of ω_t, we get

$$\omega_t = h_t b_{t-1}\frac{Y_{t-1}}{Y_t}. \tag{24}$$

Under Assumption 3, we obtain from (**??**) and (16): $Y_t/Y_{t-1} = a_t$. Then the equilibrium conditions (17) to (19) reduce to the following set of equations:

$$q_t = \beta E_t\left(\frac{h_{t+1}}{a_{t+1}}\right), \tag{25}$$

$$q_t b_t = h_t\frac{b_{t-1}}{a_t} + g - \hat{\tau}, \tag{26}$$

and:

$$h_t = \begin{cases} \mathbf{h}\cdot a_t\omega_t^{\max}/b_{t-1} < 1 & \text{if } b_{t-1}/\omega_t^{\max} > a_t, \\ 1 & \text{if not.} \end{cases} \tag{27}$$

Notice that, taking the sequence $\{\omega_t^{\max}\}$ as given, these three equations are sufficient to analyze both the pricing of public debt, q_t, and the dynamics of emitted debt-to-output ratio, b_t, in the constrained fiscal regime.

4.1 Pricing public debt.

In order to solve the model consisting of equations (25) to (27), it helps to make the following conjecture:

Conjecture 1. $\omega_{t+1}^{\max}$ *is known in* t.

In the sequel we will restrict the analysis to a class of equilibria for which this conjecture holds.

Let us denote by δ_t the ratio of b_t to $\omega_{t+1}^{\max}$,that is: $\delta_t \equiv b_t/\omega_{t+1}^{\max}$. Since we suppose than $\omega_{t+1}^{\max}$ is known in t, the default condition in $t+1$ ($B_t > \Omega_{t+1}^{\max}$) can be rewritten:

$$b_t = \frac{B_t}{Y_t} > \frac{\Omega_{t+1}^{\max}}{Y_t} = a_{t+1}\omega_{t+1}^{\max}, \tag{28}$$

or, more simply:

$$\delta_t > a_{t+1}.$$

Then, using (27) for $t+1$, and using the probability distribution of a_{t+1}, the price of a public bond, q_t, given by (25) can be rewritten as:

$$q_t = \tilde{q}\left(\delta_t, \mathbf{h}\right) \equiv \beta \begin{cases} E\left(1/a\right) & \forall \delta_t \leq a_{\inf}, \\ \mathbf{h}G\left(\delta_t\right)/\delta_t + \int_{\delta_t}\left(1/a\right)dG\left(a\right) & \forall \delta_t \in \left(a_{\inf}, a^{\sup}\right), \\ \mathbf{h}/\delta_t & \forall \delta_t \geq a^{\sup}. \end{cases} \tag{29}$$

This equation defines three regions:

1. When δ_t is very low (less than $a_{\inf}$), that is, when the emitted debt-to-output ratio is low relative to the future default threshold *per* unit of future GDP, there is no risk of default even in the worse situation (when the shock is very adverse). Therefore the default-risk premium is nil and q_t is equal to $\beta E\left(1/a\right)$. This value determines the risk-free gross interest rate $1/\beta E\left(1/a\right)$ which is lower than β^{-1}, reflecting the cost to be paid by agents (in terms of yield loss) for the insurance offered by public debt as a risk-free investment in this risky economy.

2. When δ_t is in an intermediate range which happens to be $(a_{\inf}, a^{\sup})$, then the prospect of default cannot be discarded and the default-risk premium is no more nil. It grows with b_t, and therefore the price decreases with it.

3. When δ_t is very high (above $a^{\sup}$), what happens for high values of b_t relative to $\omega_{t+1}^{\max}$, default is certain. Therefore the price of the sovereign bond is more strongly decreasing with δ_t and more directly related to the debt recovery rule parameter $\mathbf{h}$.

To resume, except in the region of no default, the price of the sovereign bond is a decreasing function of δ_t, that is, a decreasing function of the debt-to-output ratio b_t, and an increasing function of the future default threshold relative to future GDP $\omega_{t+1}^{\max}$.

Let us now define $v_t \equiv q_t b_t$, the market value of debt relative to output. From (29), and using again $\delta_t = b_t/\omega_{t+1}^{\max}$, we get:

$$v_t = \chi\left(\delta_t, \mathbf{h}\right) \cdot \omega_{t+1}^{\max}, \tag{30}$$

with

$$\chi\left(\delta_t, \mathbf{h}\right) \equiv \tilde{q}\left(\delta_t, \mathbf{h}\right) \delta_t \tag{31}$$

Using (29) and (31) we can rewrite (30) on a more explicit form:

$$v_t = \begin{cases} \beta E\left(1/a\right) b_t & \forall b_t \leq a_{\inf}\omega_{t+1}^{\max}, \\ \beta \mathbf{h}\omega_{t+1}^{\max} \cdot G\left(\frac{b_t}{\omega_{t+1}^{\max}}\right) + \beta \int\limits_{b_t/\omega_{t+1}^{\max}} \left(\frac{b_t}{a_{t+1}}\right) dG\left(a_{t+1}\right) & \forall b_t \in \left(a_{\inf}\omega_{t+1}^{\max}, a^{\sup}\omega_{t+1}^{\max}\right), \\ \beta \mathbf{h}\omega_{t+1}^{\max} & \forall b_t \geq a^{\sup}\omega_{t+1}^{\max}. \end{cases} \tag{32}$$

We refer to this last equation as the public debt valuation equation. It relates the market value of debt (relative to output) to the amount of public debt (relative to output) supplied by the government. Like the pricing equation (29), it permits to define three different regions:

1. When b_t is very low (less than $a_{\inf}\omega_{t+1}^{\max}$), there is no risk of default and the value of emitted public debt is simply the quantity of bonds discounted at the risk-free gross interest rate.

2. When b_t is in an intermediate range which happens to be $\left(a_{\inf}\omega_{t+1}^{\max}, a^{\sup}\omega_{t+1}^{\max}\right)$, the price of a bond is a decreasing function of the emitted quantity of these bonds. Therefore the public debt value, $v_t \equiv q_t b_t$, is potentially non-monotone in b_t.

3. When b_t is very high (above $a^{\sup}\omega_{t+1}^{\max}$), default is certain. Therefore the value of sovereign bonds (in level) is the discounted value of debt after rescheduling:

   $$E_t Q_{t,t+1} \cdot h_{t+1} B_t = E_t \beta \frac{Y_t}{Y_{t+1}} \cdot \mathbf{h}\Omega_{t+1}^{\max} = \beta \mathbf{h}\omega_{t+1}^{\max} Y_t,$$

   giving the result in equation (32): $v_t = \beta \mathbf{h}\omega_{t+1}^{\max}$.

We are now able to offer the following proposition about the maximum public debt value in t:

Proposition 1. *Given $\omega_{t+1}^{\max}$, under Assumption 1, there exists a unique value $v_t^{\max}$, given by:*

$$v_t^{\max} = x_{\mathbf{h}}\omega_{t+1}^{\max}, \tag{33}$$

It is a linear function of the future default threshold. $x_{\mathbf{h}} = \chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)$ is an increasing function of $\mathbf{h}$, satisfying $x_{\mathbf{h}} < \beta$ for $0 \leq \mathbf{h} < 1$, $x_0 > 0$ and $x_1 = \beta$. Moreover, $\delta_{\mathbf{h}}$ –the value of δ which maximizes $\chi\left(\delta, \mathbf{h}\right)$– is an increasing function of $\mathbf{h}$, satisfying $a_{\inf} < \delta_{\mathbf{h}} < a^{\sup}$ for $0 \leq \mathbf{h} < 1$, and $\delta_1 = a^{\sup}$.

This proposition characterizes the value function and it proves the existence of a maximum public debt value (relative to GDP) for a given value of the future default threshold ratio $\omega_{t+1}^{\max}$. Moreover it is a linear function of this ratio, which is still assumed to be known. Since, at this maximum value $v_t^{\max}$, we must have $b_t/\omega_{t+1}^{\max} = \delta_{\mathbf{h}}$, we conclude that the amount of emitted debt which maximizes the public debt value is proportional to the future default threshold ratio $\omega_{t+1}^{\max}$.

The proof provided in the appendix is based on the study of the function $\chi(\delta_t, \mathbf{h})$, the interest of which is not to depend on $\omega_{t+1}^{\max}$. The figure 2 depicts this function for different values of $\mathbf{h}$.

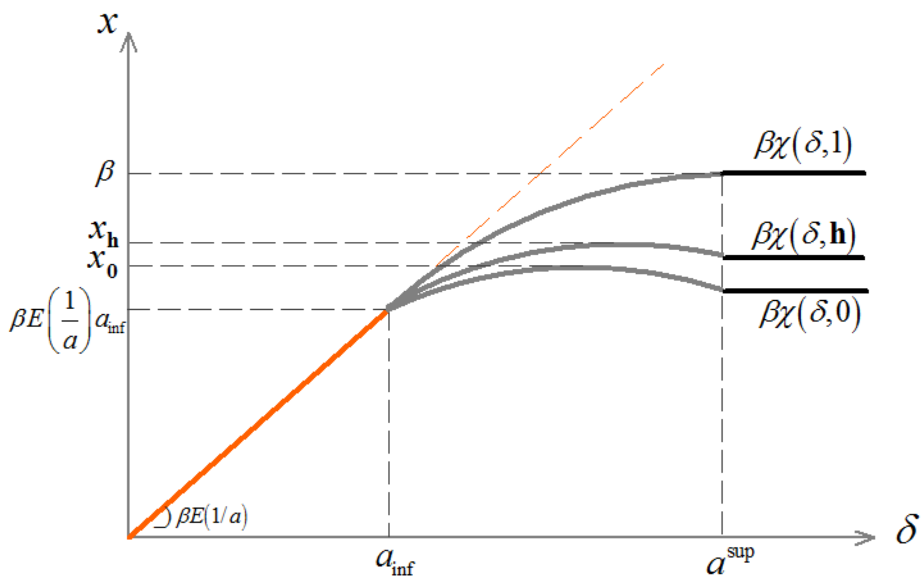

Figure 2: $\chi(\delta, \mathbf{h})$

The higher the default threshold, the higher the amount of emitted debt, $b_t = \delta_{\mathbf{h}}\omega_{t+1}^{\max}$, at which the pricing effect overcomes the direct effect of more debt and makes the public debt value, v_t, start decreasing. The higher the debt recovery ratio $\mathbf{h}$, the higher the maximal market value: Lenders are ready to lend more as they receive more in case of default. In the extreme case of no recovery ($\mathbf{h} = 0$), lenders are potentially willing to lend to the government, even in the face of possible default. They are compensated by a positive risk premium and default is not a certain event. In the extreme case of the highest recovery rate ($\mathbf{h} = 1$), the maximum public debt value is just equal to the discounted default threshold or, *per* unit of GDP:

$$v_t^{\max} = E_t Q_{t,t+1} \cdot h_{t+1} B_t / Y_t = E_t \beta \cdot \Omega_{t+1}^{\max} / Y_{t+1} = \beta\omega_{t+1}^{\max}.$$

4.2 The equilibrium default threshold.

The previous analysis allows us to clarify the notion of default as a market event and endogenize the default threshold $\omega_t^{\max}$. Let us rewrite the government budget constraint (26) by using the definition of v_t ($\equiv q_t b_t$). One find:

$$v_t = h_t \frac{b_{t-1}}{a_t} - (\hat{\tau} - g) \,. \tag{34}$$

Proposition 1 establishes the existence of a maximum for the left-hand side term of this equation, the market value of public debt emitted in t. Ruling out a strategic default decision by the fiscal authority, it does not default as long as this maximum is not reached. In other words, the no-default condition, derived from the default condition (28) applied to period t,

$$b_{t-1} \leq a_t \omega_t^{\max}, \tag{35}$$

must be equivalent to:

$$\frac{b_{t-1}}{a_t} - (\hat{\tau} - g) \leq v_t^{\max},$$

since $h_t = 1$, when sovereign default is not observed. By rewriting these two formulas as equalities and substituting for b_{t-1}, we must express the equilibrium value of $\omega_t^{\max}$, the maximum amount of public debt (per output in t) redeemable in t without default ($h_t = 1$) by means of refinancing on the market:

$$\omega_t^{\max} = v_t^{\max} + (\hat{\tau} - g) \,. \tag{36}$$

The default threshold is simply defined as the sum of the maximum value that the government can obtain from the market and the primary surplus of the period. In addition, we have proven in Proposition 1 that $v_t^{\max}$ depends on $\omega_{t+1}^{\max}$. Combining (33) and (36), we get a dynamic expression for $\omega_t^{\max}$:

$$\omega_t^{\max} = x_{\mathbf{h}} . \omega_{t+1}^{\max} + (\hat{\tau} - g) \,. \tag{37}$$

It is a forward-looking equation: How much can at most be redeemed today depends on how much can at most be redeemed tomorrow, because this last term directly determines the opportunities for public funding.

Denoting by $\omega_{\mathbf{h}}^{\max}$ the stationary value of the default threshold in equation (37), we get the following proposition:

Proposition 2. *The equilibrium default threshold as a percentage of GDP, $\omega_t^{\max}$, is locally unique and equal to:*

$$\omega_t^{\max} = \omega_{\mathbf{h}}^{\max} \equiv \frac{1-\beta}{1-x_{\mathbf{h}}} \omega^{\sup}, \forall t. \tag{38}$$

$\omega_{\mathbf{h}}^{\max}$ is an increasing function of $\mathbf{h}$. In the special case $\mathbf{h} = 1$, we obtain $x_{\mathbf{h}} = \beta$ and $\omega_{\mathbf{h}}^{\max} = \omega^{\sup}$.

Proof. using (23), (37) can be rewritten

$$\omega_t^{\max} = x_{\mathbf{h}}.\omega_{t+1}^{\max} + (1-\beta)\,\omega^{\sup},$$

whose stationary value is given by equation (38). From Proposition 1, $x_{\mathbf{h}} < \beta$, $\forall$ $\mathbf{h} < 1$, and $x_1 = \beta$. Inspecting (38), we check that $\omega_{\mathbf{h}}^{\max} < \omega^{\sup}$, $\forall$ $\mathbf{h} < 1$, and $\omega_1^{\max} = \omega^{\sup}$. Furthermore, $x_{\mathbf{h}} < 1$, $\forall$ $\mathbf{h} \leq \mathbf{1}$, implies that the forward-looking equation (37) has an unstable dynamics around the unique stationary equilibrium, $\omega_{\mathbf{h}}^{\max}$, which is determinate and locally unique. From previous proposition, $x_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, thus $\omega_{\mathbf{h}}^{\max}$ is an increasing function of $\mathbf{h}$ too. □

We first notice that unless $x_{\mathbf{h}}$ is equal to β –its upper limit corresponding to the case $\mathbf{h} = 1$–, the default threshold[16] is lower than $\omega^{\sup}$, the solvency ratio. This proves the impact of the debt recovery rule on the limit imposed by the market on the public debt ratio, which has to fulfill the condition $B_{t-1}/Y_t < \omega_{\mathbf{h}}^{\max}$ so that default does not occur. The lower the recovery rate, $\mathbf{h}$, the lower the default threshold. Except in the case $\mathbf{h} = 1$, the upper limit of public debt-to-GDP ratio is not derived from the transversality condition applied to the intertemporal government budget constraint.

Secondly, a corollary of Proposition 2 is that the maximum debt value given by equation (33) is state-independent and given by:

$$\upsilon_t^{\max} = \upsilon_{\mathbf{h}}^{\max} \equiv x_{\mathbf{h}}\omega_{\mathbf{h}}^{\max} = (1-\beta)\,\frac{x_{\mathbf{h}}}{1-x_{\mathbf{h}}}\omega^{\sup}, \forall t,$$

where the last equality is obtained by using (38).

4.3 The equilibrium market value of public debt.

From (30), (31) and using Proposition 2, we can define a valuation function relating the market value of debt to the amount of public debt (relative to output) supplied by the government, parameterized by $\mathbf{h}$:

$$\upsilon\,(b_t;\mathbf{h}) \equiv \chi\left(\frac{b_t}{\omega_{\mathbf{h}}^{\max}},\mathbf{h}\right)\cdot\omega_{\mathbf{h}}^{\max}, \tag{39}$$

$$= \tilde{q}\left(\frac{b_t}{\omega_{\mathbf{h}}^{\max}},\mathbf{h}\right)b_t \tag{40}$$

where $\omega_{\mathbf{h}}^{\max}$ is given by (38) and $\tilde{q}\,(b_t/\omega_{\mathbf{h}}^{\max},\mathbf{h})$ is always defined by (29).

Denoting by $b_{\mathbf{h}}$, the amount of the debt-to-output ratio for which $\upsilon\,(b_t;\mathbf{h})$ reaches its maximum, $\upsilon_{\mathbf{h}}^{\max}$, that is:

$$\upsilon\,(b_{\mathbf{h}};\mathbf{h}) = \upsilon_{\mathbf{h}}^{\max}, \tag{41}$$

we can represent the curve $\upsilon\,(b;\mathbf{h})$, and the equilibrium condition (9) on a same figure. Figure 3 gives this representation for $\mathbf{h}$ <1.

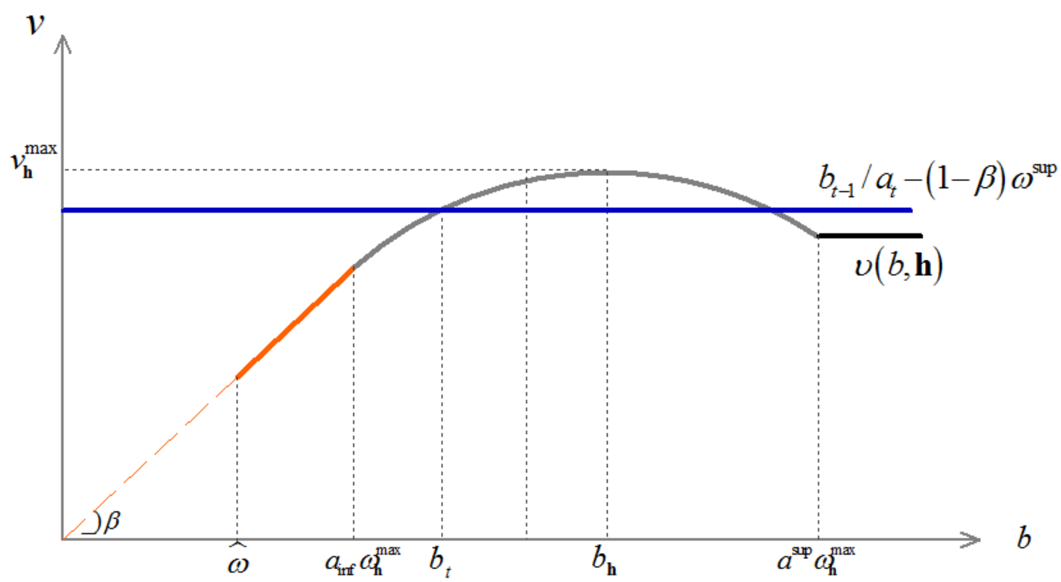

Figure 3: $\upsilon\,(b;\mathbf{h})$

This curve makes clear that $b_{\mathbf{h}}$ is the quantity of debt which generates the highest financing that the Treasury can get from the market at any date. Finally an equilibrium level of debt is such that it satisfies both (39) and (34). The first equation corresponds to the valuation function and is represented by the non-linear curve displaying three different shapes over the three intervals which we uncovered in section 4.1 (see 32), the second corresponds to the government budget constraint and is represented by the horizontal straight line. There are two values of b_t which meet this request.

We can observe that the equilibrium situated on the decreasing side of the valuation function is "unstable" in the Walrasian sense. [17] Thus a stability argument leads to the selection of the low debt equilibrium, satisfying $b_t \leq b_{\mathbf{h}}$. Taking into account the possibility of default, the equilibrium debt-to-output ratio is then given by:

$$b_t = \min b \text{ such that } \upsilon\,(b;\mathbf{h}) = -\,(1-\beta)\,\omega^{\sup} + \begin{cases} b_{t-1}/a_t & \text{if } b_{t-1}/a_t \leq \omega_{\mathbf{h}}^{\max}, \\ \mathbf{h}\cdot\omega_{\mathbf{h}}^{\max} & \text{if not.} \end{cases} \tag{42}$$

This equation allows us to uncover the role of the current shock. If the debt due in t (b_{t-1}) is lower than $a_t\omega_{\mathbf{h}}^{\max}$, there is no default. There is default if it is higher. In other words, for a given level of debt, a high enough supply shock allows the government

[16]We do not need now to specify which default threshold we refer to as the stationary and the equilibrium ones are the same.

[17]In the neighborhood of the high debt equilibrium, in the case of an excess demand an increase in the price of bonds increases the gap between demand and supply; the reverse is true in the case of an excess supply. MW use a similar argument to select the lower equilibrium.

to find on the market the ways to finance its deficit and not default: A large shock (a high value of a_t) generates high output, and reduces the debt burden. Graphically speaking, a large shock lessens the horizontal line in 3. This makes clear that default is provoked by adverse current conditions, given the structural parameters of the economy, including the debt recovery ratio $\mathbf{b}$.

5 Public debt dynamics and sustainability.

We are now equipped to turn to the issue of public debt sustainability and study the dynamics of public debt when default triggered by financial markets is taken into account. In this section we propose a new approach to public debt sustainability, grounded on market behavior and the existence of default as defined above.

We have just seen in the previous section that default is triggered by sufficiently bad circumstances (a low enough value of a_t). We also know from the forward-looking nature of the economy that the anticipation of a future default may lead to default. This is clearly seen in the exposition of the model in section 2 where we reasoned contingently on the future default threshold. It therefore implies that it is the whole random future which

This suggests to address the following two questions:

1. suppose that at a given date t, it is forecast that in the indefinite future the shock realizations be "bad" enough. Does it imply that a sovereign default is unavoidable in the future?

2. suppose to the contrary that at a given date t, it is forecast that in the indefinite future the shock realizations be "good" enough. Does it imply that a sovereign default is never to happen in the future?

To answer these two questions, we offer the following

Definition 1. *A public debt is said to be "$\underline{\varphi}-$sustainable" at date t when its trajectory does not reach the default threshold at any future date, assuming that there is no growth rate realization a_{t+s} lower than $\underline{\varphi} \leq 1$.*

Definition 2. *A public debt is said to be "$\overline{\varphi}-$unsustainable" at date t when its trajectory reaches the default threshold at some finite date, assuming that there is no growth rate realization a_{t+s} higher than $\overline{\varphi} \geq 1$.*

The answer to the first question is "no" when we prove that the public debt is *"$\underline{\varphi}-$sustainable"*; the answer to the second question is also "no" when when we prove that the public debt is *"$\overline{\varphi}-$unsustainable"*. If we are able to show under which circumstances a public debt is *"$\underline{\varphi}-$sustainable"*, we conclude in these circumstances the sustainability of public debt (the avoidance of default) is possible without having ruled out the possibility of default. Similarly, if we are able to prove under which circumstances a public debt is $\overline{\varphi}-$*unsustainable,* we may conclude that even very rosy expectations of the future prospects of an economy may not be sufficient to prevent default.

This approach to public debt sustainability differs from the standard approach. Macroeconomists are used to think, in compliance with common sense, that public debt is either sustainable or not. Using the single criterion of the transversality condition this is natural: either this condition is met or not. But the canonical approach is not based on a market-driven theory of sovereign default and does not reason in a consistent stochastic environment. When we take into consideration the presence of persistent shocks, the dynamics of public debt is more complex and the dilemmas on public debt sustainability more elaborate.

As seen above the default is triggered on the market when lenders are unwilling to buy public bonds. Therefore the occurrence of default and the implied (un-)sustainability of public debt relies on the dynamics of the market pricing of sovereign debt. Formally we need to concentrate on the market value function. Replacing υ_t with $\upsilon(b_t; \mathbf{h})$ in (34):

$$\upsilon(b_t; \mathbf{h}) = h_t b_{t-1}/a_t - (1-\beta)\,\omega^{\text{sup}} \tag{43}$$

where $\upsilon(b_t; \mathbf{h})$ is given by (39). Together with $h_t = \mathcal{H}\left(b_{t-1}/a_t, \omega_{\mathbf{h}}^{\max}\right)$, this equation makes clear that there is a stochastic dynamic process linking the succeeding amounts of emitted debt. In order to answer the two above questions, we need to focus on this equation.

5.1 "$\varphi-$Risky steady state"

First we reason for a given φ and generalize the notion of a risky steady state by offering the notion of a $\varphi-$risky steady state. We denote by $b_{\mathbf{h}}^*(\varphi)$ the stationary level of b_t in equation (43) with $h_t = 1$, $a_t = \varphi$, $\forall t$. It is the solution of the following equation:

$$\upsilon(b; \mathbf{h}) = \varphi^{-1} b - (1-\beta)\,\omega^{\text{sup}} \tag{44}$$

where $\upsilon(b; \mathbf{h})$ is given by (39). $b_{\mathbf{h}}^*(\varphi)$ is a stationary level of debt when growth rate realization a_{t+s} is equal to φ at all s.

A special case is $\varphi = 1$. The growth rate realization a_{t+s} equal to $\varphi = 1$ at all s. This corresponds to the study of a "risky steady state". $b_{\mathbf{h}}^*(1)$ then corresponds to a RSS level of debt (see literature).

Generalizing this notion, we refer to $b_{\mathbf{h}}^*(\varphi)$ as the "$\varphi-$RSS level of debt". This concept is a necessary step for making sense of the two notions of of $\underline{\varphi}-$sustainable and $\overline{\varphi}-$unsustainable as will be clear in the next subsection. Regarding this level we offer the following

Proposition 3. *In the critical fiscal regime ($\tau_t = \hat{\tau}$), for a given debt reduction rule (i.e. a given* $\mathbf{h}$*) there exists a pair* $\varphi_{\inf}$ and $\varphi_{\mathbf{h}}^{\sup}$ *satisfying* $a_{\inf} \leq \varphi_{\inf} < 1 < \varphi_{\mathbf{h}}^{\sup} \leq a^{\sup}$ *such that, for any* $\varphi \in \left]\varphi_{\inf}, \varphi_{\mathbf{h}}^{\sup}\right[$,

1. there exists a unique level $b_{\mathbf{h}}^*(\varphi)$ satisfying (44),

2. $b^*_{\mathbf{h}}(\varphi)$ is increasing in φ,

3. $\hat{\omega} < b^*_{\mathbf{h}}(\varphi) < \varphi\omega^{\max}_{\mathbf{h}} < b_{\mathbf{h}}$, for $\mathbf{h} < 1$ and any $\varphi \in \left]\varphi_{\inf}, \varphi^{\sup}_{\mathbf{h}}\right[$.

This proposition makes clear that the two notions of $\underline{\varphi}-$sustainable and $\overline{\varphi}-$unsustainable are consistent with our setting as the interval for which a $\varphi-$RSS level of debt can be defined belongs to the existing interval of shock realizations. Moreover the $\varphi-$RSS level of debt for any admissible value of φ is below the maximum amount of emitted debt. As could be expected, the higher is the value of the shock under consideration (φ) the higher is the corresponding RSS level of debt.

5.2 $\varphi-$Dynamics

We now study the dynamics of public debt and the impact of the supply shocks on the prospect of default and therefore the (un-)sustainability of public debt. We derive from (43) the following dynamic equation for b_t by inverting the $\upsilon(\cdot)$ function on the support $[\hat{\omega}, b_{\mathbf{h}}]$ where it is monotonously increasing and continuous (and therefore invertible):

$$b_t = \upsilon^{-1}\left(\varphi^{-1} b_{t-1} - (1-\beta)\,\omega^{\sup}\right)$$

The gray curve represents this function over the relevant support $[\hat{\omega}, b_{\mathbf{h}}]$. It is not linear because the $\upsilon(\cdot)$ itself is not linear. The $\varphi-$RSS level of debt $b^*_{\mathbf{h}}(\varphi)$ corresponds to the intersection of this curve with the 45°$-$line. It is an unstable "source".

The returning point corresponds to $b_{t-1} = \varphi\omega^{\max}_{\mathbf{h}}$. This comes from (36) and (44).

Given this instability result, and using the definitions given above, we offer the following

Proposition 4. *For any* $\left(\underline{\varphi}, \overline{\varphi}\right)$ *such that* $\varphi_{\inf} \le \underline{\varphi} \le 1 \le \overline{\varphi} \le \varphi^{\sup}_{\mathbf{h}}$,

1. *The public debt to be redeemed at* t*,* B_t*, is* "$\underline{\varphi}$-sustainable" if $b_{t-1} \equiv B_t/Y_{t-1} < b^*_{\mathbf{h}}\left(\underline{\varphi}\right)$.

2. *The public debt to be redeemed at* t*,* B_t*, is* "$\overline{\varphi}$-unsustainable" if $b_{t-1} \equiv B_t/Y_{t-1} > b^*_{\mathbf{h}}(\overline{\varphi})$.

We refer to $b^*_{\mathbf{h}}\left(\underline{\varphi}\right)$ as the "sustainability threshold", and $b^*_{\mathbf{h}}(\overline{\varphi})$ as the "unsustainability threshold".

The proof of this proposition is straightforward. If all future shocks are equal to $\underline{\varphi}$, and $b_{t-1} < b^*_{\mathbf{h}}\left(\underline{\varphi}\right)$, b_t will converge toward $\overline{\omega}$ given that $b^*_{\mathbf{h}}\left(\underline{\varphi}\right)$ is a source. Therefore if some future are above $\underline{\varphi}$, but none below, as some future amount of debt will be lower, b_t will still converge toward $\overline{\omega}$. Similarly, if all shocks are equal to $\overline{\varphi}$ and $b_{t-1} > b^*_{\mathbf{h}}(\overline{\varphi})$,

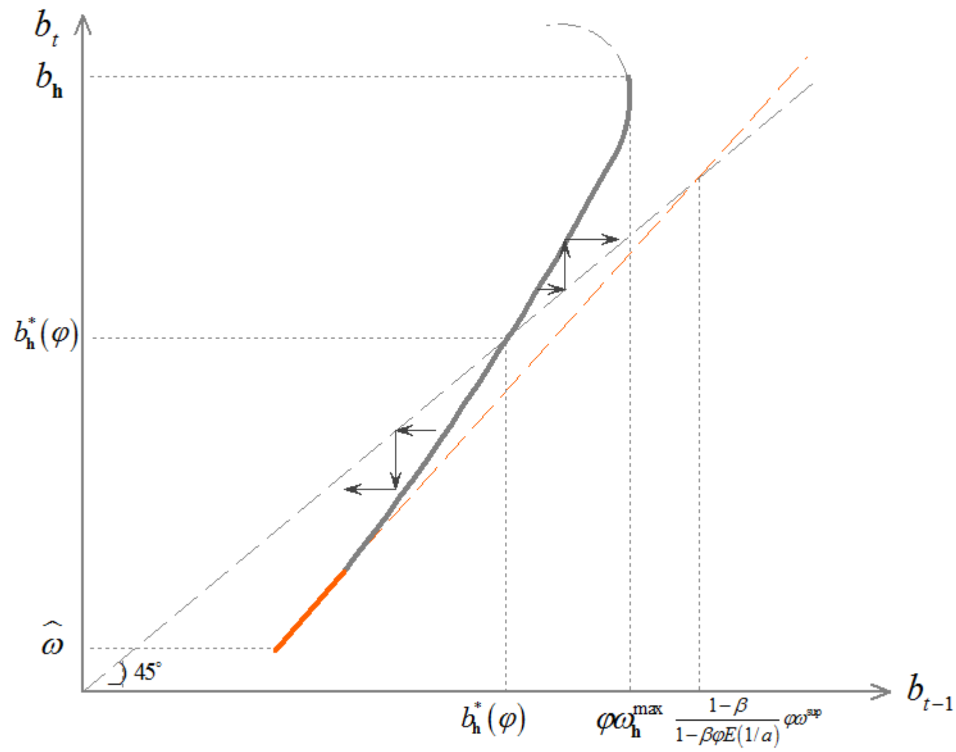

Figure 4: The $\varphi-$Dynamics

the debt converges to $\overline{\varphi}\omega^{\max}_{\mathbf{h}}$ and thus there will be default. If some future are below $\overline{\varphi}$, but none above it, as some future amount of debt will be higher, b_t will still converge toward to the default threshold.

It is now clear that if $\overline{\varphi} = \underline{\varphi} = 1$, then the two thresholds are confounded. If $b_{t-1} < b^*_{\mathbf{h}}(\varphi)$ it is $1-$ sustainable. If not, it is $1-$ unsustainable.

But if $\underline{\varphi} < 1 < \overline{\varphi}$, then the two thresholds are not confounded. Then there exists an intermediate interval between these thresholds for which it is impossible to state under which circumstances (under which assumption on shocks) the public debt will be sustainable or not. If b_{t-1} is in this interval, a positive sequence of above-average shocks may decrease it so that, at some future date, it will pass under the sustainability threshold and therefore be $\underline{\varphi}-$sustainable. Or, to the contrary, a negative sequence of below-average shocks may increase it so that, at some future date, it will pass above the unsustainability threshold and therefore be $\overline{\varphi}-$unsustainable.

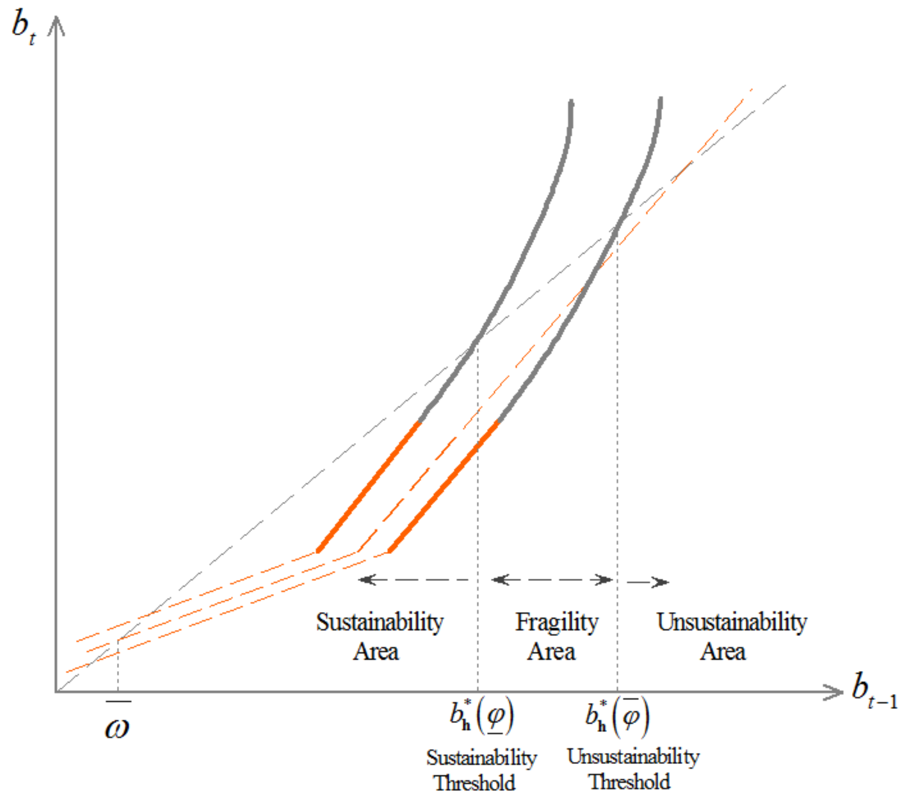

Figure 5: The Debt-to-GDP Area

5.3 Debt recovery rule and sovereign default dynamics.

In this section we focus on the impact of the debt recovery rule on the dynamics of public debt when sovereign default is not ruled out. As said in the introduction, defaults occur too late, when the debt is already too high, and the debt reduction is too little. Thus the defaulting country is still enmeshed in difficulties and is still in rough waters.

The debt recovery rule matters on the whole dynamics of public debt because it impacts on the default threshold as was proven in section 4.2. The issue now is how it impacts on the sustainabilility of public debt.

Suppose default has just occurred and public debt is reduced according to the DRR. As we have seen above, it has an ambiguous effect on the pricing of debt, and thus on the sustainability of debt through the snowball effect. A low debt recovery ratio means a large loss in the case of default and thus increases the risk premium leading to a higher burden on the (post-default) debt. But it has the positive feature of making the post-default debt very much lower from the default threshold and thus gives more room to accommodate future (negative) shocks.

This leads us to ask whether the sustainability of debt is secured, after default, by a low or a high debt recovery ratio. We answer this question in the following

Proposition 5. *1. For a given $\underline{\varphi}$ such that $\varphi_{\inf} \leq \underline{\varphi} \leq 1$, there exists a critical value $\mathbf{H}\left(\underline{\varphi}\right)$ satisfying* $0<\mathbf{H}\left(\underline{\varphi}\right)<\underline{\varphi}$ *and implicitly defined by:*

$$\mathbf{H}\left(\underline{\varphi}\right)\omega^{\max}_{\mathbf{H}(\underline{\varphi})} = b^{*}_{\mathbf{H}(\underline{\varphi})}\left(\underline{\varphi}\right),$$

such that, in case of default, the post-default debt-to-GDP ratio $\mathbf{h}\omega^{\max}_{\mathbf{h}}$ is $\underline{\varphi}$-sustainable (i.e. satisfies: $\mathbf{h}\omega^{\max}_{\mathbf{h}} < b^{*}_{\mathbf{h}}\left(\underline{\varphi}\right)$) if and only if $\mathbf{h} \leq \mathbf{H}\left(\underline{\varphi}\right)$.

2. $\mathbf{H}\left(\underline{\varphi}\right)$ is an increasing function of $\underline{\varphi}$.

According to this proposition, once default occurs, the debt recovery ratio must be sufficiently low (i.e. the debt reduction high enough) to put the post-default debt below the $\underline{\varphi}-$sustainabilility ratio and thus ensures that the post-default debt is sustainable. This result is the analytical counterpart of the empirical observation that the post-default should avoid be "too little".

This is justified by the fact that successive defaults will imply a series of debt reductions which in discounted terms will be more costly than a up-front large reduction which on the whole will not be so costly insofar at it puts the sovereign debt on a sustainable track, and avoids future defaults (contingent on some assumptions on future shocks).

An additional result is that it the future debt is to sustainable under more depressing circumstances (accommodating lower values of shocks, i.e. corresponding to a lower $\underline{\varphi}$), the debt recovery rate should be lower. In other words, the more secure (with respect to future shocks) the post-default public debt, the lower it must be.

6 Conclusion.

The issue of the compatibility of public debt sustainability with sovereign default is the concern of the present paper. The current definition of public debt sustainability, based on the sole transversality condition, rules out default. Yet it is unsatisfactory as it assumes away without proper justification the possibility of such defaults. Actually a sovereign default episode is clear evidence that the amount of public debt could not be redeemed or rolled over and thus public debt was not sustainable. Moreover, after default and an agreement with its lenders, the sovereign is able to resume borrowing and is not condemned to eternal financial autarky: there is (financial) life after default and public debt is not wiped out forever. Therefore in a macro perspective, the dynamics of public debt must be assessed when defaults episodes and the likelihood of default are taken into account.

We tackle this issue within a macro stochastic (general equilibrium) model which allows for infrequent defaults and encompasses a debt recovery rule which defines the post-default initial public debt. There are limits to the capacity to raise taxes when confronted with an increase in public spending: there is an upper limit to the tax rate which can be imposed on the economy. This creates a kink in the dynamics of expected public debt. Above this level, the financing of further public expenditures is obtained solely though borrowing and the debt burden increases more rapidly than below this threshold. The kink has the important property of generating two steady-states, only one being stable. The unstable steady-state is traditionnally associated with the upper limit for sustainable public debt.

We prove that the maximum public debt to be issued and traded on the market, which corresponds to the "default threshold", is actually below this upper limit. This is due to the fact that lenders take into consideration the prospect of future defaults and therefore include a risk premium which weighs on the financial burden of the sovereign. This maximum public debt decreases with the haircut ratio implicitly specified by the debt recovery rule which applies after default. Three factors contribute to this threshold: fiscal policy and the capacity to adjust taxes to the spending needs of the sovereign, the distribution law of shocks and, last but not least, the debt recovery rule itself.

We are then able to disentangle the role of the macro shocks and the debt recovery rule.

Turning first to the role of shocks, positive (above-average) shocks on the growth shock alleviate the burden of public debt; yet they may be insufficient given the amount of public debt to avoid default. The servicing of this debt makes it balloon and thus tend toward the default threshold. On the other hand, negative (below-average) shocks reduce the fiscal inflows and thus deteriorate public debt; however if the public debt is sufficiently low, given the low service of this debt, it tends dynamically to the stable steady-state.

Hence since shocks play a crucial role in the dynamics of public debt, the notion of public debt sustainability must be adapted so as to explicitly take into account this impact. We offer a new approach to public debt sustainability which leads us to define two thresholds: the ($\underline{\varphi}-$) sustainability threshold, and the ($\overline{\varphi}-$) unsustainability threshold. Consistent with what we said on the default threshold, this approach differs with from the standard theory of public debt sustainability based on the transversality condition. Our definitions of the thresholds explicitly rely on some assumptions on the shocks.

As about the debt recovery rule we prove that the debt recovery ratio must be sufficiently low to ensure, in the case of default, that the post-default debt is below the ($\underline{\varphi}-$) sustainabilility threshold and thus is ($\underline{\varphi}-$) sustainable.

Our analysis offers a new perspective on debt sustainability, consistent with the reality of default. Public debt is "sustainable" when, in the case of default, for a given expected future shocks and given the debt recovery rule, post-default debt is expected to converge to a stable steady-state, thus avoiding further defaults.

An implication of this is that the debt recovery rule impacts on the entire dynamics of public debt through forward-looking attitudes. It contributes to the dynamics toward default. It is thus critical to reflect on the characteristics of this rule from a macroeconomic perspective. Actually, it raises a intriguing dilemma:

1. A small amount of recovered debt (a large haircut) is desirable as, in case of

default, it ensures that the post-default debt is within the "sustainability zone": it will dynamically converge to a stable steady state, with fiscal policy ease.

2. However it has the negative consequence of increasing the likelihood of default. It reduces the maximum amount of debt which can be marketed / sold on market, and therefore reduces the capacity of the fiscal authority to deal with shocks. In other words it makes default more likely.

Hence the debt recovery ratio (identically, the haircut ratio) raises a trade-off: If too low, shock-triggered defaults occur frequently; if too high, the economy experiences serial defaults even in the absence of shocks. This leads us to think that a normative analysis of the debt recovery rule is enticing. This is beyond the scope of the present paper, we leave its exploration to further research.

Finally, we assumed away the analysis of the influence of monetary policy on the road to default and thus on the sustainability of public debt by assuming that the monetary policy maker perfectly controls the price level and applies a zero inflation rule. Putting monetary policy back in the analysis of public debt is also left for further research.[18]

[18] Schabert (2010) explores the role of monetary policy in Uribe's model and highlights the role of monetary policy in sovereign defaults.

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A Appendix

A.1 Proof of proposition 1

Let us define $\delta_t \equiv b_t/\omega_{t+1}^{\max}$. From (32) we can rewrite $\tilde{v}\left(b_t, \omega_{t+1}^{\max}\right)$ as:

$$\tilde{v}\left(b_t, \omega_{t+1}^{\max}\right) = \beta x_t \omega_{t+1}^{\max}, \tag{45}$$

with

$$x_t = \begin{cases} \delta_t & \forall \delta_t \leq a_{\text{inf}}, \\ \chi\left(\delta_t, \mathbf{h}\right) & \forall \delta_t \in \left(a_{\text{inf}}, a^{\text{sup}}\right), \\ \mathbf{h} & \forall \delta_t \geq a^{\text{sup}}, \end{cases} \tag{46}$$

where the function $\chi\left(\delta, \mathbf{h}\right)$ is such that:

$$\chi\left(\delta_t, \mathbf{h}\right) = E\left(1/a\right)\delta_t - \int^{\delta_t} \left(\delta_t/a - \mathbf{h}\right) \cdot dG\left(a\right). \tag{47}$$

The derivative of $\chi\left(\delta, \mathbf{h}\right)$ with respect to δ is:

$$\frac{\partial \chi\left(\delta, \mathbf{h}\right)}{\partial \delta} = E\left(1/a\right) - \int^{\delta} \frac{1}{a} dG\left(a\right) - \left(1 - \mathbf{h}\right) g\left(\delta\right) \equiv \Phi\left(\delta, \mathbf{h}\right). \tag{48}$$

Suppose that there exists a value $\delta_{\mathbf{h}} \in \left(a_{\text{inf}}, a^{\text{sup}}\right)$ such that:

$$\Phi\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = 0, \tag{49}$$

Then, using (47) and (49), $\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)$ can be written:

$$\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = \mathbf{h} G\left(\delta_{\mathbf{h}}\right) + \left(1 - \mathbf{h}\right) \delta_{\mathbf{h}} g\left(\delta_{\mathbf{h}}\right) \equiv x_{\mathbf{h}}. \tag{50}$$

We search for the derivatives of $\Phi\left(\delta, \mathbf{h}\right)$. We get, $\forall\ \delta \in \left(a_{\text{inf}}, a^{\text{sup}}\right)$:

$$\Phi_1\left(\delta, \mathbf{h}\right) \equiv \frac{\partial \Phi\left(\delta, \mathbf{h}\right)}{\partial \delta} = -\left[1 + \left(1 - \mathbf{h}\right) \frac{\delta g'\left(\delta\right)}{g\left(\delta\right)}\right] \left(\frac{1}{\delta}\right) g\left(\delta\right). \tag{51}$$

It is negative if and only if:

$$\frac{\delta g'\left(\delta\right)}{g\left(\delta\right)} > -\frac{1}{1 - \mathbf{h}}.$$

This condition is satisfied for any $\mathbf{h} \in (0, 1)$ if the elasticity of the probability-density function is higher than -1 (Assumption *1*). This implies:

$$\Phi_1\left(\delta, \mathbf{h}\right) < 0.$$

From (48), defining $\Phi_2\left(\delta, \mathbf{h}\right) \equiv \frac{\partial \Phi(\delta, \mathbf{h})}{\partial \mathbf{h}}$, we get

$$\Phi_2\left(\delta, \mathbf{h}\right) = g\left(\delta\right) > 0, \tag{52}$$

For $\delta_{\mathbf{h}} \in (a_{\inf}, a^{\sup})$, we have:

$$\frac{\partial \delta_{\mathbf{h}}}{\partial \mathbf{h}} = -\frac{\Phi_2(\delta_{\mathbf{h}}, \mathbf{h})}{\Phi_1(\delta_{\mathbf{h}}, \mathbf{h})} > 0. \tag{53}$$

Looking for the values $\underline{\mathbf{h}}$ and $\overline{\mathbf{h}}$ such that $\delta_{\underline{\mathbf{h}}} = a_{\inf}$ and $\delta_{\overline{\mathbf{h}}} = a^{\sup}$, we find from (48) and (49):

$$\begin{aligned} \underline{\mathbf{h}} &= 1 - \frac{E(1/a)}{g(a_{\inf})} \\ \overline{\mathbf{h}} &= 1. \end{aligned}$$

As it is assumed that $E(a) = 1$ (Assumption 1), by the Jensen Inequality $E(1/a) > 1$ and $\underline{\mathbf{h}} < 0$. As $\mathbf{h} \geq 0$, this value is irrelevant.

When $\mathbf{h}$ =0, we get from (50):

$$x_0 = \delta_0 g(\delta_0)$$

with δ_0 given by (48) and (49) when $\mathbf{h} = 0$, or equivalently:

$$E \int_{\delta_0} \frac{1}{a} dG(a) = g(\delta_0)$$

As it implies that δ_0 is positive, using Assumption 1, x_0 is strictly positive.

When $\mathbf{h}$ =1, we get:

$$x_1 = G(\delta_1)$$

with δ_1 given by (48) and (49) when $\mathbf{h} = 1$, or equivalently:

$$E\left(\frac{1}{a}\right) = \int^{\delta_1} \frac{1}{a} dG(a)$$

implying $\delta_1 = a^{\sup}$ and therefore $x_1 = 1$.

We now prove that $\chi(\delta_{\mathbf{h}}, \mathbf{h})$ is increasing in $\mathbf{h}$. From (47)and (49), for $\delta_{\mathbf{h}} \in (a_{\inf}, a^{\sup})$, we get:

$$\frac{d\chi(\delta_{\mathbf{h}}, \mathbf{h})}{d\mathbf{h}} = \frac{\partial \chi(\delta_{\mathbf{h}}, \mathbf{h})}{\partial \mathbf{h}} = G(\delta_{\mathbf{h}}) > 0. \tag{54}$$

In sum, $x_{\mathbf{h}} = \chi(\delta_{\mathbf{h}}, \mathbf{h})$ is increasing in $\mathbf{h}$ and such that: $x_0 > 0$ and $x_1 = 1$.

Henceforth the function $\chi(\delta, \mathbf{h})$ reaches a unique maximum for a value $\delta_{\mathbf{h}}$ which is at most equal to $a^{\sup}$. We then obtain the representation given in figure 2

A.2 Proof of proposition 2

From Proposition 1, $x_{\mathbf{h}} < 1$, $\forall$ $\mathbf{h} < 1$, and $x_1 = 1$. Inspecting (38), we check that $\omega_{\mathbf{h}}^{\max} < \omega^{\sup}$, $\forall$ $\mathbf{h} < 1$, and $\omega_1^{\max} = \omega^{\sup}$. Furthermore, $\beta x_{\mathbf{h}} < 1$, $\forall$ $\mathbf{h}$, implies that the forward-looking equation (37) has an unstable dynamics around the unique stationary equilibrium, $\omega_{\mathbf{h}}^{\max}$, which is determinate and locally unique. From previous proposition, $x_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, thus $\omega_{\mathbf{h}}^{\max}$ is an increasing function of $\mathbf{h}$.

A.3 Proof of proposition 3

1. We represent graphically the RHS and LHS of (44). The RHS is represented for 3 values of φ.

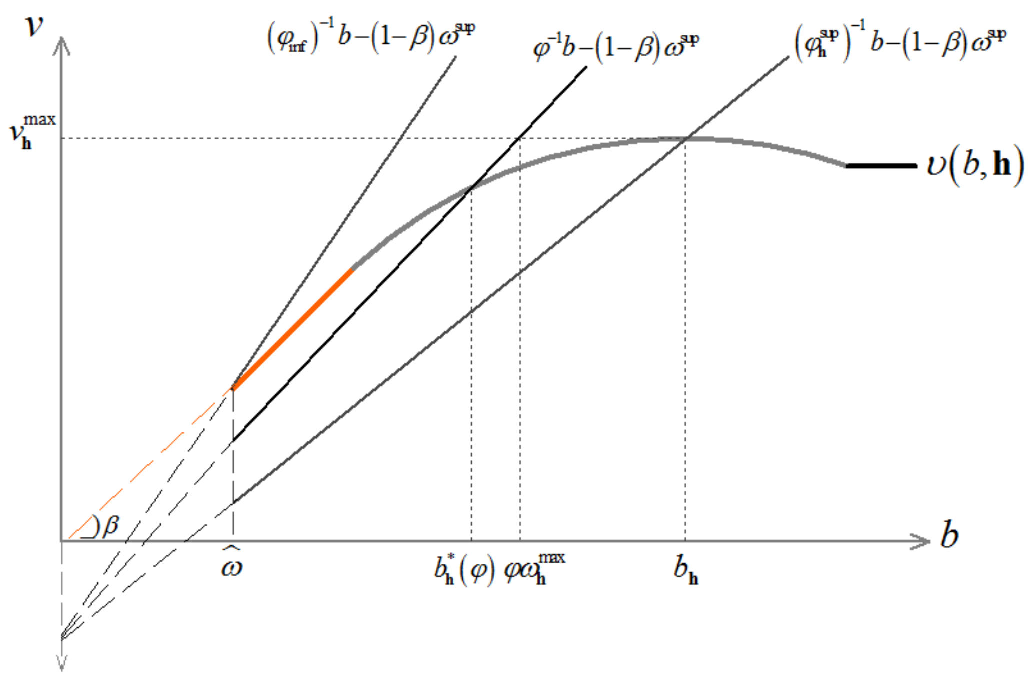

Figure 6: $\varphi-$Risky steady state

A/ A preliminary is to investigate two extreme cases of φ.

As we restrict the analysis to the fiscal limit regime, the lowest admissible value of the $\varphi-$RSS is at least equal to $\hat{\omega}$. Denote by $\overleftrightarrow{\varphi}$ the value of φ such that: $\upsilon(\hat{\omega};\mathbf{h}) = \varphi^{-1}\hat{\omega} - (1-\beta)\omega^{\sup}$. From (39) and (44) $\overleftrightarrow{\varphi}$ is such that:

$$\overleftrightarrow{\varphi} = \frac{1}{\beta E(1/a) + (1-\beta)\omega^{\sup}/\hat{\omega}}.$$

From the Jensen inequality, $E(1/a_t) > 1/E(a_t)$. As we assume that $E(a_t) = 1$, then $E(1/a_t) > 1$. Given that $\omega^{\sup}/\hat{\omega} > 1$, these two inequalities imply that $\overleftrightarrow{\varphi} < 1$.

As it may that $\overleftrightarrow{\varphi}$ is lower than $a_{\inf}$, the lowest value of $\varphi_{\inf}$ for which there is a $\varphi-$RSS in the fiscal limit regime is

$$\varphi_{\inf} \equiv max\left(\overleftrightarrow{\varphi}, a_{\inf}\right). \tag{55}$$

The second extreme case is when $b_{\mathbf{h}}$ is the solution to (44). In other words, $\varphi_{\mathbf{h}}^{\sup}$ is the (constant) realization of the growth rate such that the $\varphi-$RSS is at the highest possible market value of public debt corresponding to the DRR ratio $\mathbf{h}$. Denoting by $\varphi_{\mathbf{h}}^{\sup}$ the value of φ for which$\upsilon\left(b_{\mathbf{h}};\mathbf{h}\right) = \left(\varphi_{\mathbf{h}}^{\sup}\right)^{-1} b_{\mathbf{h}} - (1-\beta)\,\omega^{\sup}$, we get:

$$\varphi_{\mathbf{h}}^{\sup} = \frac{b_{\mathbf{h}}}{\upsilon\left(b_{\mathbf{h}};\mathbf{h}\right) + (1-\beta)\,\omega^{\sup}}$$

where $\upsilon\left(b_{\mathbf{h}};\mathbf{h}\right)$ is given by the intermediate formula in (39). In other words, $b_{\mathbf{h}} = b_{\mathbf{h}}^{*}\left(\varphi_{\mathbf{h}}^{\sup}\right)$. Given that $b_{\mathbf{h}} = \delta_{\mathbf{h}}\omega_{\mathbf{h}}^{\max}$, (38), (**??**) and (41) this is equivalent to:

$$\varphi_{\mathbf{h}}^{\sup} = \frac{\delta_{\mathbf{h}}\omega_{\mathbf{h}}^{\max}}{\beta x_{\mathbf{h}}.\omega_{\mathbf{h}}^{\max} + \left(1-\beta x_{\mathbf{h}}\right)\omega_{\mathbf{h}}^{\max}}$$

Hence $\varphi_{\mathbf{h}}^{\sup} = \delta_{\mathbf{h}}$. For $\mathbf{h} < 1$, $\delta_{\mathbf{h}} < a^{\sup}$, hence we get: $\varphi_{\mathbf{h}}^{\sup} < a^{\sup}$. For $\mathbf{h} = 1$, we get : $\varphi_{1}^{\sup} = a^{\sup}$. Equivalently $\varphi_{1}^{\sup} = a^{\sup} > \varphi_{\mathbf{h}}^{\sup} > \varphi_{0}^{\sup} > 1$ for $\mathbf{h} < 1$.

We study $\varphi_{0}^{\sup} = \delta_{0}^{\#}$. From (48) and (49) $\varphi_{0}^{\sup}$ is such:

$$E\left(1/a\right) = \int^{\varphi_{0}^{\sup}} \frac{1}{a} dG\left(a\right) + g\left(\varphi_{0}^{\sup}\right)$$

that is:

$$g\left(\varphi_{0}^{\sup}\right) = \int_{\varphi_{0}^{\sup}}^{a^{\sup}} \frac{1}{a} g\left(a\right) da. \tag{56}$$

Let us study $\check{\mathbf{h}}$ such that $\varphi_{\check{\mathbf{h}}}^{\sup} = 1$. It is such that

$$\int_{1} \frac{1}{a} dG\left(a\right) - \left(1-\check{\mathbf{h}}\right) g\left(1\right) = 0$$

hence:

$$\check{\mathbf{h}} = 1 - \frac{\int_{1} \frac{1}{a} dG\left(a\right)}{g\left(1\right)}$$

From Assumption 1.c, we know that:

$$\frac{1}{a} g\left(a\right) > -g'\left(a\right), \forall a,$$

which implies:

$$\int_{1} \frac{1}{a} g\left(a\right) da > -\int_{1} g'\left(a\right) da = \left[g\left(1\right) - g\left(a^{\sup}\right)\right]$$

From Assumption 1.b $\left(\lim_{a\to a^{\sup}} g(a)=0\right)$, therefore

$$\int_1 \frac{1}{a} g(a)\, da > g(1)$$

It follows that $\check{\mathbf{h}} < 1$. As we know from lemma 1 that $\varphi_{\mathbf{h}}^{\sup}$ is an increasing function of $\mathbf{h}$, $\varphi_{\mathbf{h}}^{\sup} > 1, \forall \mathbf{h} \geq 0$.

B/ The RHS of (44) for a given $\varphi_{\mathbf{h}}$ is represented by a straight line of slope $(\varphi_{\mathbf{h}})^{-1}$ which is necessarily lower than the straight line for $\varphi_{\inf}$, of slope $(\varphi_{\inf})^{-1}$, and above the straight line for $\varphi_{\mathbf{h}}^{\sup}$, of slope $(\varphi_{\mathbf{h}}^{\sup})^{-1}$. Given the curve representing $\upsilon(b;\mathbf{h})$, there exists an intersection between this line and this curve. Given the characteristics of $\varphi^{\sup}$ and $\varphi_{\inf}$, for any value $\varphi_{\inf} < \varphi < \varphi_{\mathbf{h}}^{\sup}$, this intersection corresponds to a unique value $b_{\mathbf{h}}^{*}(\varphi)$ satisfying (44) and such that $\hat{\omega} < b_{\mathbf{h}}^{*}(\varphi) < b_{\mathbf{h}}$.

2. Furthermore $b_{\mathbf{h}}^{*}(\varphi)$ is an increasing function of φ given that the RHS of (44) is decreasing in φ.

3. For any $\varphi \in]\varphi_{\inf}, \varphi_{\mathbf{h}}^{\sup}[$, $\hat{\omega} < b_{\mathbf{h}}^{*}(\varphi) < \varphi\omega_{\mathbf{h}}^{\max} < b_{\mathbf{h}}$ (see figure above).

A.4 Proof of proposition 5

In order to prove this proposition, we have to show that $\Psi(\mathbf{h},\varphi) \equiv \mathbf{h}\omega_{\mathbf{h}}^{\max} - b_{\mathbf{h}}^{*}(\varphi)$ is monotonously increasing in $\mathbf{h}$, with $\Psi(\mathbf{H}(\varphi),\varphi) = 0$ for a value of $\mathbf{H}(\varphi)$ such that $0 < \mathbf{H}(\varphi) < \varphi$. We define the function $\psi(\mathbf{h},\varphi)$ such that:

$$\psi(\mathbf{h},\varphi) \equiv \frac{\Psi(\mathbf{h},\varphi)}{\omega_{\mathbf{h}}^{\max}} = \mathbf{h} - \delta_{\mathbf{h}}^{*}(\varphi) \tag{57}$$

where $\delta_{\mathbf{h}}^{*}(\varphi) \equiv b_{\mathbf{h}}^{*}(\varphi)/\omega_{\mathbf{h}}^{\max}$. Since $\omega_{\mathbf{h}}^{\max} > 0$, a sufficient condition to have *1.* is that the function $\psi(\mathbf{h},\varphi)$ be a function continuously increasing in $\mathbf{h}$, $\forall\mathbf{h}$ such that $0 \leq \mathbf{h} \leq \varphi$, or equivalently $\psi(0,\varphi) < \psi(\mathbf{H}(\varphi),\varphi) = 0 < \psi(\varphi,\varphi)$.

By differentiating $\psi(\mathbf{h},\varphi)$, we find:

$$\frac{\partial\psi(\mathbf{h},\varphi)}{\partial\mathbf{h}} = 1 - \frac{\partial\delta_{\mathbf{h}}^{*}(\varphi)}{\partial\mathbf{h}}. \tag{58}$$

We know from (44) and (38) that:

$$\frac{\upsilon(b_{\mathbf{h}}^{*};\mathbf{h})}{\omega_{\mathbf{h}}^{\max}} = \varphi^{-1}\delta_{\mathbf{h}}^{*}(\varphi) - \left(1 - \beta x_{\mathbf{h}}^{\#}\right)$$

From (45) and (46), for $\delta_t \in (a_{\inf}, a^{\sup})$, we get*:*

$$\delta_{\mathbf{h}}^{*}(\varphi) = 1 - \beta\left[\chi(\delta_{\mathbf{h}},\mathbf{h}) - \varphi\chi(\delta_{\mathbf{h}}^{*}(\varphi),\mathbf{h})\right] \tag{59}$$

which allows us to get, using (49) and (**??**):

$$\frac{\partial\delta_{\mathbf{h}}^{*}(\varphi)}{\partial\mathbf{h}} = \beta\left(\frac{\varphi\frac{\partial\chi\left(\delta_{\mathbf{h}}^{*}(\varphi),\mathbf{h}\right)}{\partial\mathbf{h}} - \frac{\partial\chi(\delta_{\mathbf{h}},\mathbf{h})}{\partial\mathbf{h}}}{1 - \beta\varphi\frac{\partial\chi\left(\delta_{\mathbf{h}}^{*}(\varphi),\mathbf{h}\right)}{\partial\delta_{\mathbf{h}}^{*}(\varphi)}}\right),$$

Using (47) we get:

$$\frac{\partial \delta_{\mathbf{h}}^*(\varphi)}{\partial \mathbf{h}} = \beta \left(\frac{\varphi G(\delta_{\mathbf{h}}^*(\varphi)) - G(\delta_{\mathbf{h}})}{1 - \beta\varphi \frac{\partial \chi\left(\delta_{\mathbf{h}}^*(\varphi), \mathbf{h}\right)}{\partial \delta_{\mathbf{h}}^*(\varphi)}} \right)$$

From (48) and assumption 1, we have $\beta\varphi\partial\chi(\delta_{\mathbf{h}}^*(\varphi), \mathbf{h}) / (\partial\delta_{\mathbf{h}}^*(\varphi)) < 1$, and the denominator is positive. From proposition 44, as $b_{\mathbf{h}}^*(\varphi) \leq b_{\mathbf{h}}$, the numerator is negative. Hence $\partial\delta_{\mathbf{h}}^*(\varphi) / \partial\mathbf{h}$ is negative. Given the continuity properties of these functions, $\psi(\mathbf{h}, \varphi)$ be a function continuously increasing in $\mathbf{h}$.

By computing $\psi(0, \varphi)$ and $\psi(\varphi, \varphi)$, we get:

$$\psi(0, \varphi) = -\delta_0^*(\varphi) < 0,$$

and

$$\psi(\varphi, \varphi) = \varphi - \delta_\varphi^*(\varphi) > 0,$$

as $\delta_\varphi^*(\varphi) \equiv b_\varphi^*(\varphi) / \omega_\varphi^{\max} < \varphi < 1$ from Proposition 2 and the imposed restriction on φ. Therefore there exists a value $0 < \mathbf{H}(\varphi) < \varphi$ such that $\psi(\mathbf{H}(\varphi), \varphi) = 0$, or equivalently $\Psi(\mathbf{H}(\varphi), \varphi) = 0$.

2. We study:

$$\frac{\partial \mathbf{H}(\varphi)}{\partial \varphi} = -\frac{\psi_\varphi(\mathbf{H}(\varphi), \varphi)}{\psi_{\mathbf{h}}(\mathbf{H}(\varphi), \varphi)}$$

The denominator is positive as shown above. From (59) we get:

$$\frac{\partial \delta_{\mathbf{h}}^*(\varphi)}{\partial \varphi} = \frac{\beta\chi(\delta_{\mathbf{h}}^*(\varphi), \mathbf{h})}{1 - \beta\varphi \frac{\partial \chi\left(\delta_{\mathbf{h}}^*(\varphi), \mathbf{h}\right)}{\partial \delta_{\mathbf{h}}^*(\varphi)}} > 0$$

and therefore the denominator is positive. Thus $\partial\mathbf{H}(\varphi) / \partial\varphi$ is positive.