Tuning in RBC Growth Spectra[*]

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Abstract

The paper explains an array of RBC puzzles by adding to the standard RBC model external margins for both physical capital and human capital, and examining model fit with US data across business cycle (BC), low frequency (LF), and Medium Cycle (MC) windows. The model results in a goods sector productivity shock with a 7500 times smaller variance than the standard RBC model, implying greatly improved shock amplification and an enhanced explanation of a wide array of correlations, volatilities and growth persistence across the windows. The model matches the data cyclicality of the main shares of GDP and GDI such as a countercyclic consumption-output ratio, procyclic investment-output ratio, countercyclic labor share of income and countercyclic capital depreciation share of income. Also matched is a countercyclic human capital investment time, a procyclic capacity utilization rate, and the declining output growth persistence autocorrelation profile that is known as the "propagation" puzzle. Using a distance metric and a uniform grid search, measures of fit are presented by window and category. In the BC window, key correlations have an average 15% deviation from the data moments; the LF growth persistence has an average 8% deviation from the data moments.

Keywords: Real Business Cycle, amplification, growth persistence, data moments, human capital, utilization rate, low frequency.

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1 Introduction

Real business cycle (RBC) models face many puzzles, such as matching the fundamental consumption-output ratio data, improving shock amplification, matching output growth persistence, and matching labor data. The challenge includes explaining the unusual US post-2008 data of a deep recession and prolonged, low frequency, below-trend, recovery. One approach is to step back to the overview of RBC problems as given by King & Rebelo (2000). They reveal how adding external margins for labor and capital (variable utilization rates) allows a reduction in the variance of the total factor productivity (TFP) shock. They argue that adding external margins provides better amplification with "much smaller shocks", that these shocks can be measured in a way similar to the backing out of shocks in a model with home production by Ingram et al. (1997), and that explaining low frequency fluctuation is important.

This paper contributes a cohesive improvement on an array of puzzles across frequencies by adding the two external margins through a variable utilization rate of both human capital and physical capital. Adding a human capital investment sector enables a variable utilization rate of human capital as determined by leisure, while also enabling endogenous growth and permanent income effects from shocks. Including a productivity shock to the human capital sector as well as to the goods sector, the paper exploits the method of backing out the shocks from data series by additionally demanding that the model calibration has the same variance-covariance matrix as the backed-out productivity shocks (Benk et al., 2005; see also King and Rebelo, 2000). From the three extensions of having both external margins for both human and physical capital, of including a human capital sectoral shock within endogenous growth, and of forcing the model to be consistent with the backed-out shock properties, the first startling result is a huge "amplification" of the goods sector TFP productivity shock. Rather than the dynamics of the model being driven by a large variation in the TFP shock, with a coincident weak shock amplification, instead the goods sector productivity shock has a variance 7500 times smaller than the 0.007 King & Rebelo (2000) standard, implying strong shock amplification.[1]

The human capital investment sector productivity shock has a mean some 25 times less than the goods sector productivity shock, and a three times smaller variance, giving it a 22000 times smaller variance than the standard RBC TFP shock variance of 0.007. This small but potent shock to the growth rate of human capital creates a permanent income effect that does not overwhelm the goods sector temporary income TFP shock, but instead supplements it so as to allow the latter to be greatly reduced. In effect, rather than founding the model dynamics upon only a temporary TFP effect, the adding of the human capital sector shock with a near-one correlation with the goods sector TFP shock boosts the

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[1] In comparison, King & Rebelo (2000) report a reduction from 0.007 to 0.001 from adding external margins.

temporary income shock with just enough of a permanent income effect as needed to explain the basic moments at the heart of the RBC challenges, including the consumption-output puzzle. Consumption varies by more because it follows permanent income which in turn now also rises by more; this is a result of the permanent income hypothesis of consumption holding within such Ramsey-based models as in this paper. Using the backed-out shocks and US data, the paper demonstrates graphically how the model explains the US historical consumption-output ratio in comparison to the data.

The paper's results include a full array of standard RBC comparisons of model moments to data moments across correlations, volatilities and growth persistence, and across not only the business cycle window (BC), but also the low frequency (LF) and the Medium Cycle (MC) windows (Comin & Gertler 2006). In addition, the paper presents a matching of moment correlations with output for a set of the "great ratios" that correspond to the main categories of our National Income and Product Accounts, as shares of GDP and GDI. Unlike standard RBC results, the paper establishes a broad match at the BC and lower frequency windows on the GDP side for the countercyclic consumption to output and procyclic investment to output ratios, these being the most well-known great ratios and the ones that directly relate to the RBC consumption-output puzzle by which consumption does not vary in the model as much as in the data. As in GDI data, the labor share of income is counter-cyclical in the BC window, as in Hansen & Prescott (2005). At the BC and lower frequency windows, as in data, the capital share of income is procyclic, the capital depreciation share of income is counter-cyclical[2] and the physical capital utilization rate is procyclic. The human capital investment time is countercyclical at the BC and lower frequency windows, for which Dellas & Sakellaris (2003) provide supporting business cycle evidence.

Additional results are that the paper profers a solution of the "propagation" puzzle, being the explanation of the autocorrelation profiles of the growth rate of output, consumption and investment as found in data (Benhabib et al. 2006). There is also evidence of asymmetry in the business cycle, from a Cooley et al. (1995), Harding & Pagan (2002) and Hansen & Prescott (2005) perspective, in that the model's backed-out TFP shock has smaller magnitude recessions than expansions. Other notable puzzle features are that: an initial slightly negative labor impulse is found, as has been reported in data and in theory (Benhabib et al. 2006, Gali 1999); and there exists a Chari et al. (2007)-related labor wedge that is explained using human capital time in a way similar to McGrattan's (2015) explanation using intangible capital investment time.

The paper quantifies results through contribution of a measure of model fit of the data moments. Extending Jermann (1998), the distance metric provides an aggregate average

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[2] Matched in data to the NIPA: Net Operating Surplus as a share of GDI, Fixed capital consumption as share of GDI, respectively; the authors are not aware of research explaining these share facts, or the utilization rate facts.

fractional deviation of the model's simulated moments from the data's moments for any given target set. This quantifies the degree of success in the moment comparison, which has been used since Kydland & Prescott (1982) and Long & Plosser (1983). The metric is used analytically to facilitate the calibration choice by allowing a focus on the calibration sets with the lowest metrics, and then to guage dimensions of model performance. This metric can be applied as a measure of fit to any model.

The metric results are presented in aggregate and broken down within each of the four frequencies (high, business cycle, low and Medium), as well as separately for each correlations, volatilities and growth persistence. For example, results are presented in the business cycle window in which there is an average of only a 15 percent deviation of the model moments from the data moments. In the low frequency window, growth persistence shows an even smaller 8 percent deviation from data. For the broadest set of 67 targets, which includes the high frequency window, there is an average 46 percent deviation of model moments from data moments.

Following Nolan & Thoenissen (2009), the paper also demonstrates how the backed-out goods sector TFP shock compares to the standard Solow residual, as presented in both unfiltered form and in the BC, LF and MC windows. While seeming to track well the historical US Solow residual up until 2010, after that the Solow residual falls continuously while the model's TFP shock instead begins rising as does actual US GDP growth; also as in post-2010 GDP growth data, the model's backed out shock recovery remains below trend. This model performance may reflect the King & Rebelo (2000) and McGrattan (2015) criticism that the Solow residual is not an exact measure of the economy's TFP and that instead the smaller shocks of a two sector economy may offer improvement on a TFP measure,[3] with better amplification, propagation, and moment matching.

2 Related Literature

Relative to the seminal literature on external margins, the paper's external labor margin builds upon Hansen (1985), Rogerson (1988), Benhabib et al. (1991), Greenwood & Hercowitz (1991), Lucas (1988), Perli (1998) and Gomme and Rupert (2007). The paper's variable physical capital utilization rate builds upon the one-sector RBC models of Cooley et al. (1995), Hansen & Prescott (2005), and Greenwood et al. (1988) by extension to two sectors and by making the depreciation rate a function of the utilization rate as in the functional form found in DeJong et al. (1996), Greenwood et al. (1988) and Benhabib & Wen (2004); except unlike Dejong et al. 1996, the utilization rate of physical capital is the same across sectors and there is only one sector producing an investment good, the human capital

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[3] King & Rebelo (2000) call the smaller resulting residuals when also using a second, home, production sector the "Crucini residuals" (in their footnote 60); McGrattan (2015) finds adding a second investment sector "quantitatively important for analyzing U.S. aggregate fluctuations".

sector, which in turn differs for example from the home production sector of the two-sector Gomme and Rupert (2007) approach.

King et al. (1988) extend cyclic analysis to growth spectra at the low frequency, as does Comin & Gertler (2006); this paper uses a simulation methodology consistent with the former, and a human capital focus not found in the latter. Kehoe & Prescott (2007) help explain depressions with RBC productivity shocks; and Hansen & Ohanian (2016) extend the RBC model to multiple sectors for explaining low frequency data; this paper differs from the latter two by using two sectors, as opposed to one-sector or numerous sector models respectively. Buera & Moll (2016) use financial frictions and heterogeneous agents to explain RBC "wedges"; this paper extends a representative agent approach while explaining the labor wedge with human capital as related to McGrattan & Prescott (2014) explanation using intangible capital.

Cogley & Nason (1995), Rotemberg & Woodford (1996), Perli & Sakellaris (1998) and Benhabib et al. (2006) highlight the weak internal propagation of the standard RBC model in terms of matching the data profile of a falling output growth persistence. Benhabib et al. (2006) match this profile by adding additional physical capital investment sectors with independent shocks; at the same time they find an initially negative labor impulse response in support of Gali (1999), who in turn argues that data is consistent with a negative labor TFP response rather than the standard RBC positive labor TFP response.[4] This paper achieves better propagation, along with an initially negative (slightly) labor impulse, in a way similar to Perli & Sakellaris (1998) who add a human capital investment sector for improved propagation. Like DeJong & Ingram (2001), who use a human capital investment sector to model a countercyclical human capital investment time, as supported by evidence consistent with Perli & Sakellaris (1998) and Dellas & Sakellaris (2003), the paper uses this sector in a similar way but without including as additional investment sectors as in DeJong & Ingram (2001).

McGrattan (2015) uses multiple output sectors and shocks, and an economy-wide TFP shock; Hansen & Ohanian (2016) use correlated sectoral shocks but not a separate investment sector; and Benhabib & Wen (2004) uses demand shocks. Abstracting from the numerous sector specific shocks of those contributions, this paper is closer to Maffezzoli (2000) in using only two, productivity-only, sectoral shocks, of the goods and human capital investment sectors. However this paper uses a less restricted covariance-variance matrix that breaks the mold of identical shocks that Maffezzoli (2000) uses to good effect. The backing out of shocks extends **?**.

Grossman et al. (2016) focus on long term model properties with human capital, but without application to data. Christiano et al. (2001) solves basic RBC puzzles, including the equity risk premium that this paper does not address, but this paper instead uses homothetic

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[4] See Benhabib et al. (2006) Figures 1 and 5 for the match of output growth's autocorrelation profile and Figure 7 for their generation of a Gali (1999) type labor impulse response.

utility and production without an adjustment cost of the physical capital stock. The paper's inclusion of the "great ratios" moment comparison reflects the original Klein & Kosobud (1961) focus, which has had some attention in the RBC literature, such as on the wage share of output by Cooley et al. (1995) and Hansen & Prescott (2005).

Section 3 describes the full model and its balanced growth path features. Section 4 describes the calibration, the backed-out shocks and impulse responses. Section 5 presents moment results; Section 6 details the distance metric and its results; Section 7 discusses the results more broadly; and Section 8 concludes.

3 The Model

The best-case, minimalist, "nesting" model for accomplishing the simultaneous puzzle-tasks is the general model with the two external margins, the so-named Model 2. Results are compared to a special case called Model 1 that sets the physical capital utilization rate equal to one. Model 1 does reasonably well except for matching the growth rate autocorrelation profiles, known as the propagation puzzle. Model 1 also has less shock amplification with only a 46 times smaller goods sector TFP shock variance, as compared to 0.007, rather than the 7500 times decrease of Model 2. Selected results are also presented for a standard RBC model without either external margin (which can be specified as a special case of Model 2 in which the human capital growth rate exogenous). This case uses the King and Rebelo (2000) calibration) but falls short of Model 2 on both volatility and propagation moments.

For the general model, the representative agent time t utility $U(t)$ depends on consumption, c_t, leisure, x_t, and a function of the utilization rate of physical capital denoted by $u_t \in [0,1)$; with $A \in R_+$, $B \in R$ and $\sigma \in R_+$, the time t period utility is given by

$$U(c_t, x_t, u_t) = \frac{\left[c_t x_t^A (1-u_t)^B\right]^{1-\sigma} - 1}{1-\sigma}, \tag{1}$$

which enables the existence of a balanced growth path (BGP) equilibrium.[5] Or if all productive time is denoted by l_t, and the time endowment is 1, then the utility is equally written as $U(c_t, x_t, u_t) = \frac{\left[c_t(1-l_t)^A(1-u_t)^B\right]^{1-\sigma}-1}{1-\sigma}$, which makes each $1-l_t \equiv x_t$ and $1-u_t$ the "non-utilization" rates of human and physical capital, respectively.

The only difference from standard models in which physical capital utilization enters the model is that here the utility function is allowed to depend not only on the rate at which human capital is not utilized, which is leisure in such models as this, but also upon the rate at which physical capital is not utilized. This creates a symmetry in modeling how both the growth rate of the economy depends on the capacity utilization rate of each of the two

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[5] For more, see King et al. (1988).

capitals, human and physical capital, and the utility depends upon the non-utilization rate of each human and physical capital. While including one minus the fraction of productive labor time, which equals leisure, is a standard addition to the utility function, including the one minus the productively used physical capital is not standard, but it creates an otherwise missing aspect of the symmetry in DeJong et al. (1996).

Since the model below makes the depreciation rate of physical capital an increasing function of the physical capital utilization rate, it already adds convexity into the cost of using physical capital as a factor of production. So whether the agent ultimately has a negative or positive sign in utility through the sign of B is left to the interaction of both preferences and the increased convexity through the depreciation rate. A negative sign tempors the depreciate rate's addition to convexity while a positive sign increases the degree of convexity. The case of $B = 0$, so that u_t drops out of the utility function, is allowed as a possible calibration choice, as B can be positive or negative. In the calibration below, B ends up robustly negative (at -0.16), as consistent with the Otani (1996) "spillover" view, implying a utility gain from more fully utilizing physical capital.[6] Model 1 is specified with $(1 - u_t)^B = 1$, so u_t no longer enters utility, and elsewhere u_t is fixed at one.

The representative agent time endowment of 1 for each period t, is allocated to l_{gt}, the fraction of time spent in goods production, to l_{ht}, the fraction of time spent in human capital investment production, and to x_t, leisure:

$$1 = x_t + l_{gt} + l_{ht}. \tag{2}$$

This makes $l_t \equiv l_{gt} + l_{ht}$ the time spent productively, which is also the human capital utilization rate.

Physical capital investment, i_{kt}, determines the capital stock k_t accumulation as in DeJong et al. (1996):

$$k_{t+1} = k_t - \delta\left(u_t\right) k_t + i_{kt}, \tag{3}$$

where $\delta\left(u_t\right)$ is a function, with the form

$$\delta\left(u_t\right) = \frac{\delta_k}{\psi} u_t^{\psi}, \tag{4}$$

with $\psi > 1$ and $\delta_k > 0$. A faster rate of utilization results in a higher rate of depreciation. It follows that $\delta'\left(u\right) > 0$ and $\delta''\left(u\right) > 0$ so that the marginal cost of utilizing physical capital

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[6] "In the context of this paper, a manager's experience in learning about one component of a firm has the externality of making it easier for him to learn about another component..." (p.274, Otani, 1996).

stock is increasing in the utilization rate.

Denote by y_t the real goods output . For the goods production function A_g is a positive factor productivity parameter, z_t^g the total factor productivity shock, v_{gt} the share of the physical capital stock being allocated to the goods sector and $v_{gt}u_tk_t$ the amount of physical capital in the goods sector that is utilized for production purposes. Let h_t denote the stock of human capital at the beginning of time period t; then $l_{gt}h_t$ represents the effective labor input, or the share of human capital used in goods production. With $\phi_1 \in [0,1]$, goods production is divided between consumption c_t and investment i_{kt}, as given by

$$A_g e^{z_t^g}(v_{gt}u_tk_t)^{\phi_1}(l_{gt}h_t)^{1-\phi_1} = c_t + i_{kt}. \tag{5}$$

The human capital stock is accumulated through a production sector for investment:

$$h_{t+1} = (1-\delta_h)h_t + A_h e^{z_t^h}\left[(1-v_{gt})\,u_tk_t\right]^{\phi_2}(l_{ht}h_t)^{1-\phi_2}\,, \tag{6}$$

where $\delta_h \ \in R_{++}$ is the depreciation rate, $A_h \in R_{++}$, $e^{z_t^h}$ the sectoral productivity shock, $\phi_2 \in [0,1]$, $v_{ht} = 1 - v_{gt}$ and $v_{ht}u_tk_t$ is the amount of physical capital used in the production of human capital investment.

3.1 Shock Structure

In the economy are two random shocks following first-order autoregressive processes:

the goods productivity shock z_t^g, where

$$z_t^g = \rho_g z_{t-1}^g + \varepsilon_t^g, \quad 0 < \rho_g < 1, \tag{7}$$

and the human capital investment sector productivity shock z_t^h, where

$$z_t^h = \rho_h z_{t-1}^h + \varepsilon_t^h, \quad 0 < \rho_h < 1 \tag{8}$$

and the innovations are normally distributed according to

$$\begin{pmatrix} \varepsilon_t^g \\ \varepsilon_t^h \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma}), \tag{9}$$

where the general structure of the second-order moments is the variance-covariance matrix $\mathbf{\Sigma}$, with individual variances denoted by σ_g^2 and σ_h^2. This allows for any degree of covariance between the shocks.

The social planner's problem is

$$\max_{\{c_t, l_{gt}, l_{ht}, x_t, v_{gt}, v_{ht}, u_t, k_{t+1}, h_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{[c_t x_t^A (1-u_t)^B]^{1-\sigma} - 1}{1-\sigma}, \tag{10}$$

subject to (2)-(9).

3.2 Definition of Equilibrium

Definition 1 *A general equilibrium of this model is a set of contingent plans* $\{c_t,\ k_{t+1},\ h_{t+1},\ i_{kt},\ v_{gt},\ u_t,\ x_t,\ l_{gt},\ l_{ht}\}$ *that solve the social planner's maximization problem in (10) for the initial endowment* $\{k_0,\ h_0\}$ *and exogenous stochastic technology processes* $\{z_t^g,\ z_t^h\}$*, with initial conditions* $\{z_0^g,\ z_0^h\}$ *and variance-covariance matrix* $\mathbf{\Sigma}$*.*

Appendix A presents the equilibrium conditions.

Definition 2 *A deterministic balanced growth path (BGP) equilibrium of this model is a set of paths* $\{c_t,\ k_{t+1},\ h_{t+1},\ i_{kt},\ v_{gt},\ u_t,\ x_t,\ l_{gt},\ l_{ht}\}$ *that solve the central planner's maximization problem in (10) for the initial endowment* $\{k_0,\ h_0\}$ *and exogenous technology parameters* $\{z_t^g = 0, z_t^h = 0\}$*, such that* $\{c_t, k_t, h_t, i_{kt}\}$ *grow at a common trend, and* $\{v_g, u, x, l_g, l_h\}$ *are constant.*

Proposition 3 *The social planner equilibrium is the same as the representative agent's competitive equilibrium.*

Proof. The same equilibrium conditions result as in the social planner problem, as there are no (Lucas, 1988; Maffozolli, 2000) externalities, as in Gomme (1993). ■

3.3 Balanced Growth Path Behavior

The marginal rate of substitution between goods and leisure as usual equal to the marginal product of labor, denoted by $w_t \equiv (1-\phi_1)A_g(\frac{v_g u k_t}{l_g h_t})^{\phi_1}$, or $\frac{A}{x_t}\frac{c_t}{h_t} = w_t$. From the physical capital utilization rate equilibrium condition, and with the marginal product of capital denoted by $r_t \equiv \phi_1 A_g(\frac{v_g u k_t}{l_g h_t})^{\phi_1 - 1}$, the second BGP intratemporal margin sets to a constant the time t ratio of the value of unused capital resources. In particular the constant ratio of the respective preference parameters for lesser utilization equals the ratio of the rental value of unused human capital to the rental value (net of the change in the depreciation rate $\frac{\partial \delta_k(u_t)}{\partial u_t}$) of unused physical capital:

$$\frac{A}{B} = \frac{x_t w_t h_t}{(1-u_t)\left(r_t - \delta_k u^{\psi-1}\right) k_t}. \tag{11}$$

This enables changes in utilization rates along external margins to substitute for reallocations of resources along internal margins within sectors, as consistent with preferences on unused resources. Negative values of B can result if $\delta_k u^{\psi-1} > r$; with a greater degree of convexity reflected in a relatively higher value than standard for $\psi > 1$, as in the calibration below, it does result that B is negative.

The Euler conditions for k_t and h_t are in terms of the stationary BGP growth rate g, time preference $\beta \equiv \frac{1}{1+\rho}$, and the marginal products and the utilization rates of each respective capital. With the BGP goods sector marginal product of physical capital being stationary and given as $MP^k_{Gt} \equiv \phi_1 A_g \left(\frac{l_g h_t}{v_g u k_t}\right)^{1-\phi_1}$, and the BGP human capital investment sector marginal product of human capital being stationary and given as $MP^h_{Ht} \equiv (1-\phi_2)\, A_h \left(\frac{v_h u k_t}{l_h h_t}\right)^{\phi_2}$, the BGP intertemporal conditions are

$$1+g = \left[\frac{1+(u)\,MP^k_{Gt} - \frac{\delta_k}{\psi}u^{\psi}}{1+\rho}\right]^{1/\sigma} = \left[\frac{1+(1-x)\,MP^h_{Ht} - \delta_h}{1+\rho}\right]^{1/\sigma}. \tag{12}$$

The return on capital is equalized across sectors along the BGP such that $(u)\,MP^k_{Gt} - \frac{\delta_k}{\psi}u^{\psi} = (1-x)MP^h_{Ht} - \delta_h$, by optimal choice of the factor input ratios.

The relative price of human capital investment to goods output is denoted by p_{ht}, and defined by the ratio of the shadow value of human capital investment to physical capital investment (goods output), where $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$ and λ_t and χ_t are as given in Appendix A equations (21) and (22). Using these equilibrium conditions, the BGP relative output price can be expressed as a constant multiplied by the stationary BGP factor input ratio in either the goods or human capital investment sector respectively:

$$p_{ht} = \frac{A_g}{A_h}\left(\frac{1-\phi_1}{1-\phi_2}\right)^{1-\phi_2}\left(\frac{\phi_1}{\phi_2}\right)^{\phi_2}\left(\frac{v_g u k_t}{l_g h_t}\right)^{\phi_1-\phi_2} = \frac{A_g}{A_h}\left(\frac{1-\phi_1}{1-\phi_2}\right)^{1-\phi_1}\left(\frac{\phi_1}{\phi_2}\right)^{\phi_1}\left(\frac{v_h u k_t}{l_h h_t}\right)^{\phi_1-\phi_2} \tag{13}$$

The stationary implicit factor rental prices along the BGP in turn imply that this rental price ratio is proportional to the stationary sectoral input ratios: $\frac{w_t}{r_t} = \frac{1-\phi_1}{\phi_1}\frac{v_g u k_t}{l_g h_t} = \frac{1-\phi_2}{\phi_2}\frac{v_h u k_t}{l_h h_t}$. In price-theoretic fashion, the relative price of output along the BGP is therefore a stationary function of the implicit input rental price ratio:

$$p_{ht} = \frac{A_g}{A_h}\frac{(1-\phi_1)^{1-\phi_1}}{(1-\phi_2)^{1-\phi_2}}\frac{(\phi_1)^{\phi_1}}{(\phi_2)^{\phi_2}}\left(\frac{w_t}{r_t}\right)^{\phi_1-\phi_2}. \tag{14}$$

Because it is assumed that $\phi_1 > \phi_2$, it is true that along the BGP the relative price of human capital investment p_{ht} rises when the wage to interest rate ratio w_t/r_t rises; if in contrast, $\phi_1 = \phi_2$, then $p_{ht} = A_g/A_h$.

4 Calibration, Impulse Responses, and Backed-out Shocks

By normalizing the variables that grow along the balanced growth path (BGP) by h_t, and log-linearizing the equilibrium conditions around their normalized growth paths, a stochastic system of linear equations results that is solved in terms of the state variable k_t/h_t and the two shock processes, z_t^g and z_t^h by the method of undetermined coefficients Uhlig (1998).

4.1 Calibration

Table 1 presents the calibrated structural and exogenous shock parameters for Models 1 and 2. This section describes the Model 2 calibration; Model 1 is calibrated similarly. Table 2 presents the calibration grid ranges, in which 5,000 steps within the ranges were employed. For Model 2, there are 67 targets, of which three are BGP equilibrium values, being g, x, and u. The high target number resulted from experiments which found a better fit with more targets, but with a diminishing return to adding targets. For Model 1, there are 56 targets.

The calibration methodology of Jermann (1998) is modified and combined with the shock identification scheme of Benk et al. (2005). The quarterly data period is 1972:1 to 2015:4 for Model 2, and 1959:1 to 2015:4 for Model 1. Model 2's data period is restricted by the physical capital utilization data that begins in 1972:1. Appendix B provides the data description.

Quarterly long-run BGP targets are based on US data. Strict targets for Model 2 are the balanced growth rate of the economy, g, leisure time, x, and the physical capital depreciation rate $\frac{\delta_k}{\psi}u^{\psi}$, which following Gomme & Rupert (2007) are set at 0.0035, 0.5 and 0.025 respectively. The physical capital utilization rate, u, is set at the data value of 0.785. These imply the utility weight of leisure A through the marginal rate of substitution between goods and leisure, from Appendix A equations (16) and (17), and the utility weight B, from Appendix A equations (19) and (20); these are 1.10 and -0.159 respectively. These in turn, with Appendix A equation (22), imply the productivity parameter $A_h = 0.032$ and the depreciation rate $\delta_k = 0.19$. From this it results that $\psi = 3.34$, a relatively higher degree of convexity than in Greenwood et al. (1988) where it is 1.42; note however that for $\psi > 1$, the marginal cost of increased utilization rises as ψ rises, making it more convex at 3.34 as compared to 1.42. To calibrate the remaining seven structural parameters, a grid within a bounded parameter space is established with lower and upper bounds for parameters as set out in Table 2.[7] The net of depreciation interest rate, $r - \frac{\delta_k}{\psi}u^{\psi}$, is 0.0268 while r is 0.052.

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[7] This uses similar features to *Bayesian* estimation by setting bounds with prior information; instead of the Bayesian estimation of a parameter within each set of bounds, the grid search here computes distance

For the grid ranges of Table 2, the lower bound of the discount factor β is set to 0.95 and the upper bound to 0.99. The parameter for of the constant elasticity of substitution (CES) in utility is bounded between 0.40 and 2.00 as found for example in the quarterly estimates of Hall (1988) and Mehra & Prescott (1985) respectively. The Cobb-Douglas coefficient for physical capital in the goods producing sector, ϕ_1, has a range between 0.30 and 0.40; the Cobb-Douglas coefficient for physical capital in the human capital investment sector ϕ_2 has a range between 0.08 and 0.29, consistent with Jorgenson & Fraumeni (1991) and Jones et al. (2005). The productivity parameter of the goods sector A_g is bounded between 0.50 and 2.00; the convexity parameter ψ is bounded between 2.00 and 4.00; the human capital depreciation rate δ_h is bounded between 0.001 and 0.015, as consistent with DeJong & Ingram (2001), Jorgenson & Fraumeni (1991), and Jones et al. (2005); and the shock persistence parameters ρ_g and ρ_h are bound between 0.01 to 0.99. The the cross-correlation between the two sectoral shocks is bounded between -0.999 to 0.999, since technically the model is bounded away from -1 and 1 in order to retain a positive semi-definite variance-covariance matrix. To reduce computational intensity, the initial guess for each of the shock variances is set at 0.007 as found in King and Rebelo (2000).

For each possible combination of the grid coordinates the models are solved with iterative convergence of the backed-out shock's properties to the model's assumed shock properties, as in Benk et al. (2005). This extends the method of Jermann (1998) by iterative convergence of the shocks and a mean normalization of the distance metric to transform each individual distance measure into percentage deviations of the simulated moments from the US data targets.[8]

The metric is denoted by D_z, with $z = 1, 2$ for Models 1 and 2. It is constructed by using the simulation-based moment vector, denoted by Θ, along with the corresponding US data-based target moment vector, $\hat{\Theta}$. It is defined so as to give the average fractional deviation of the model moment from the data moment across all targeted moments. This is found by summing up each of the fractional deviations of model moment from data moment, and dividing by the total number of targeted moments; call the latter T. Then the definition of D_z is $D_z \equiv \left(\sum_i \sum_j \sum_k \left|\hat{\Theta}_{ijk} - \Theta_{ijk}\right| / \left|\hat{\Theta}_{ijk}\right|\right) \Big/ T_z$, with $i = 1, 2, 3$ for the targeted moment categories of each 1) correlations, 2) volatilities, and 3) autocorrelation lags; $j = 1, ..., 5$ represents the four band-pass filtered frequencies (HF, BC, LF, MC) plus the unfiltered data used only for the autocorrelation lags (see Persistence ** in Table 7A below); and k is a function of (i, j, z) that equals the number of targets used within each

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metrics uniformily across each bounded parameter space.

[8] The approach is alternative to use of a simulated annealing algorithm, which was also explored, but which gives a different calibration with each run because of its "temperature-gauge" property; simulated annealing is also embedded in Bayesian estimation of the calibration parameters. Complete Matlab codes of the grid search approach as well as simulated annealing, both with iterative convergence of the model shocks to data, are available with detailed descriptions upon request.

moment category (i) and data frequency (j), for each of Models 1 and 2 (z).[9]

The resulting metric is used to examine the results of the top 200 best metric (lowest measures).[10] The lowest obtained metric for Model 2 was 0.41, while the one presented in the Tables has a value of 0.46; this can be interpreted as on average a 46% deviation of the full set of 67 targets from their model-achieved values. Section 6 below reports detailed metric results. The calibration and shock construction procedure yield a 7500 times smaller shock variance for the goods sector productivity shock and 22000 times smaller for the human capital investment sector shock, as compared to the standard RBC 0.007 (King & Rebelo 2000), indicating improved amplification; see Table 1.

Parameter	Description	Model 1	Model 2
β	Discount Factor	0.972	0.986
σ	CES Parameter	0.850	0.412
A	Weight of Leisure	1.11	1.10
B	Weight of Capacity Util.	$-$	-0.159
A_g	Scale Parameter of Goods Sector	1.65.	0.80
A_h	Scale Parameter of Human Sector	0.065	0.032
ϕ_1	Physical Capital Share in Goods Production	0.319	0.36
ϕ_2	Physical Capital Share in Human Investment	0.162	0.20
δ_k	Depreciation Parameter (Physical Capital)	0.018	0.19
ψ	Convexity of Endog. Depr. Rate	$-$	3.34
δ_h	Depreciation Rate of Human Capital	0.010	0.001
ρ_g	Auto-correlation of TFP	0.98	0.98
ρ_h	Auto-correlation of Human Shock	0.99	0.98
σ_g^2	Variance of TFP	$1.52x10^{-4}$	$9.4x10^{-7}$
σ_h^2	Variance of Human Productivity Shock	$1.47x10^{-4}$	$3.2x10^{-7}$
$\sigma_{g,h}$	Correlation of Shock Innovations	0.994	0.995

Table 1: Model 1 and 2 calibration parameter values.

Parameter	Description	Grid Range	
		Model 1	Model 2
β	Discount Factor	$0.95 - 0.99$	$0.95 - 0.99$
σ	CES Parameter	$0.40 - 2.00$	$0.40 - 2.00$
A	Weight of Leisure	BGP^*	BGP^*
B	Weight of Capacity Util.	$-$	BGP^*
A_g	Scale Parameter of Goods Sector	$0.50 - 2.00$	$0.50 - 2.00$
A_h	Scale Parameter of Human Sector	BGP^*	BGP^*
ϕ_1	Physical Capital Share in Goods Production	$0.30 - 0.40$	$0.30 - 0.40$
ϕ_2	Physical Capital Share in Human Investment	$0.08 - 0.29$	$0.08 - 0.29$
δ_k	Depreciation Parameter (Physical Capital)	$0.015 - 0.030$	BGP^*
ψ	Convexity of Endog. Depr. Rate	$-$	$2.00 - 4.00$
δ_h	Depreciation Rate of Human Capital	$0.001 - 0.015$	$0.001 - 0.015$
ρ_g	Auto-correlation of TFP	$0.01 - 0.99$	$0.01 - 0.99$
ρ_h	Auto-correlation of Human Shock	$0.01 - 0.99$	$0.01 - 0.99$
σ_g^2	Variance of TFP	0.007(initial)	0.007(initial)
σ_h^2	Variance of Human Productivity Shock	0.007(initial)	0.007(initial)
$\sigma_{g,h}$	Correlation of Shock Innovations	$(-0.99) - 0.99$	$(-0.99) - 0.99$

Table 2: Model 1 and 2 grid search ranges. (* *BGP* refers to calibrated values for parameters obtained through use of BGP conditions).

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[9] Alternatively, a 0.99 correlated metric is $D_{alt} = [(\hat{\Theta} - \Theta)/\hat{\Theta}]'\Omega[(\hat{\Theta} - \Theta)/\hat{\Theta}]$, where Ω is an identity matrix of the size of the number of targets k, and D_{alt} is a squared Euclidean distance; D_z in contrast is an average fractional deviation of model from data moments. D_{alt} is of interest as it is a special case of the Mahalanobis (1936) distance.

[10] We thank Viktor Huszar, DWO LLC., for the use of a massive parallel processing system; however this procedure can be run on commercially available cloud services.

4.2 Impulse Responses

The productivity shock impulse is defined as a simultaneous 1.0 percent goods sector TFP shock increase combined with a 0.04 percent human capital sector productivity increase, reflecting the calibrated 1/25 ratio of A_g/A_h. The simultaneity used is comparable to the way in which the shocks hit the economy in simulation since their correlation is 0.995. Figures show the impulse to the model's variables for 200 periods, longer than the standard 40 used in the RBC literature, in order to see results covering the full Medium cycle, as defined from 2 to 200 periods (Comin and Gertler, 2006). They include the Model 2 results in blue and for comparison a standard RBC model's impulse response in red which is based on a King and Rebelo (2000) calibration.

Figure 1 shows the permanent income effect of the shock in Model 2 in that **Y** and **C** and **K** rise for the whole future horizon for Model 2, but drop off in the RBC model. As the cause of this, Figure 1 shows in the upper-lefthand side tile that while the growth rate of output **Gy** falls for both Model 2 and the RBC model, there is a prolonged increase in the growth rates of physical and human capital in Model 2 but not in the RBC model, as seen in the tiles with the physical capital investment rate: **Ik;** its ratio to output: **Ik/Y;** and its ratio to physical capital: **Ik/K**. For the human capital, the tiles with prolonged growth for Model 2 are the human capital investment rate: **Ih;** its ratio to the human capital stock: **Ih/H;** and the growth rate of human capital: **Gh.** For Model 2's consumption to output and physical capital investment to output ratios, **C/Y** and **ik/Y,** respectively, the ratios fall and rise by more, respectively, and do so for longer, as compared to the RBC model. This prolonged countercyclical nature of the key c/y ratio and the procyclic nature of the i_k/y ratio is consistent with respective data moment correlations with output found in US data.

Note also in Figure 1 the much smaller changes "required" of the model in relative prices and factor ratios as compared to the standard RBC model. The interest to rate ratio, **R/W**, falls slightly as compared to the RBC. And similarly the physical capital to human capital ratio in the Model 2 goods sector, **Fg** (which is defined as$\frac{v_g u k_t}{l_g h_t}$), rises only slightly as compared to the 10 fold higher rise in the RBC model's capital to labor ratio **K/Lg** (which is defined as$\frac{k}{l_g}$).

Figure 2 has two marked contrasting results of the Model 2 relative to the RBC model. The interest rate, as given by **R** in the figure, jumps up and falls gradually towards a zero change somewhat beyond the 200 periods for Model 2, while **R** starts falling before the 20th period in the RBC model. The contrast results because the return on human capital in Model 2 (the marginal product of human capital in the human capital investment sector; not shown) rises in profile just as does **R** (but about one-tenth of the magnitude) and then stays positive as it falls very gradually to zero at about 150 periods. This also explains in part why leisure falls so much more in Model 2 as compared to the RBC model and why

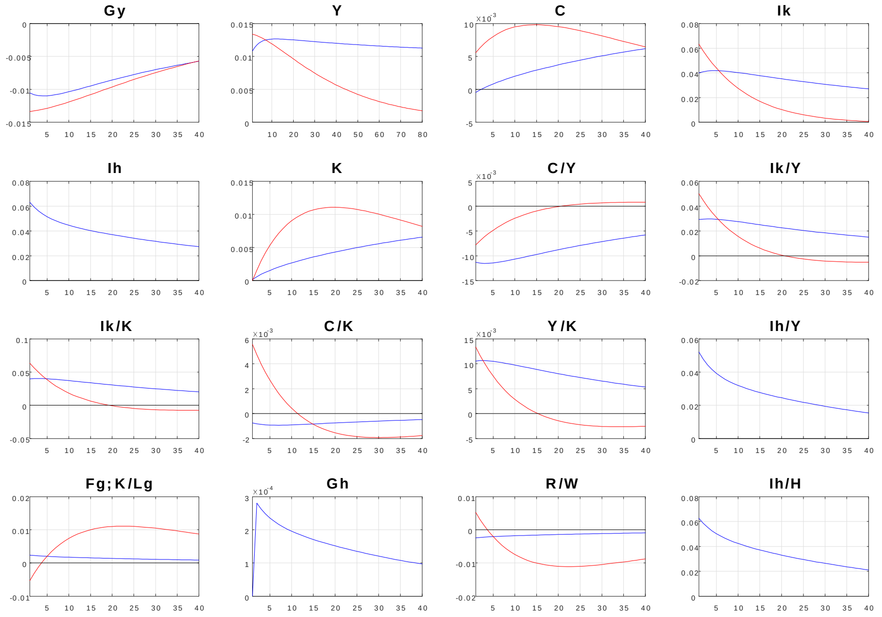

Figure 1: Effect of a Temporary Economy-Wide Shock on Output, Consumption, Ratios, and Growth Rates

time in the human capital sector, **Lh** in the figure, rises and stays positive until near to the 200th period. In addition, the result for labor time in the goods sector, **Lg** in the figure, is a nice looking (compared to data-based expectations) hump-shaped labor impulse (with a very small initial drop as in the Gali, 1999, effect). Finally the physical capital shifts to the human capital sector because both sectors are expanding, the capacity utilization rate of physical capital, **U** in the figure, jumps up and slowly falls, and so there is extra physical capital in the physical capital intensive sector, the goods sector, that can be reallocated to the human capital investment sector so as to enable both sectors to expand. This relative scarcity of human capital that induces reallocation of physical capital to the human capital investment sector is indicated by the higher relative price of human to physical capital investment, p_h which is given by **Ph** in the figure, and which jumps up and gradually falls down to zero at around 200 periods.

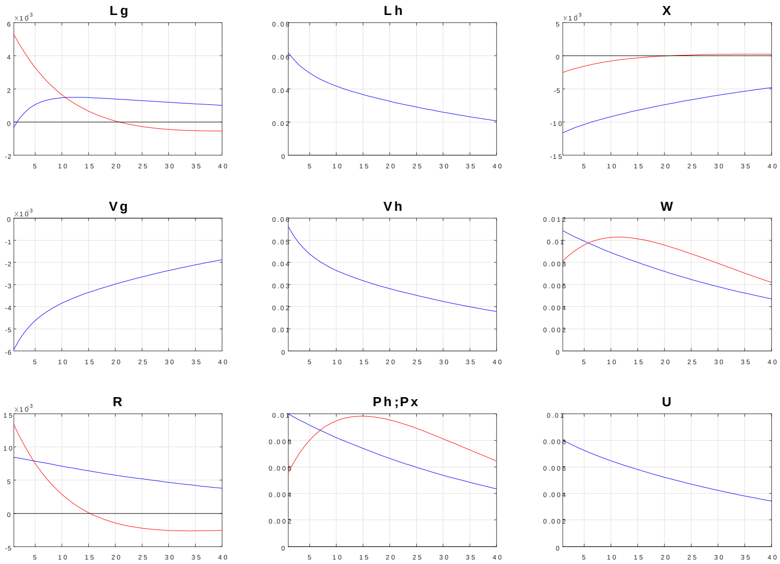

Figure 2: Effect of a Temporary Economy-Wide Shock on Sectoral Resources and Relative Prices

4.3 Estimated, Backed-out, Shocks

Figures 3 and 4 graph the unfiltered goods sector shock ("TFP") series obtained from Models 2 and 1, separately, along with the respective, traditional, Solow-residual, goods sector TFP in each graph. The model-based goods TFP shock series is constructed as in Benk et al. (2005) by matching the implicit equilibrium solution for a set of the model's decision variables to the data for each variable in that set. Note that for Model 2, seven data series are used, the consumption-output ratio, the investment-output ratio, labor hours, ratios of output, consumption, investment relative to human capital plus the utilization rate series. The Model 1 TFP is constructed with the same data set except for the utilization rate. Every quarterly data series used is for the 1972:1 to 2015:4; the starting date of 1972 is a result of using the utilization rate data which only starts then.

The method for backing-out the shocks is to match each of the chosen data series to each of the model's respective variable solutions, in terms of the known parameters, the state variable (k_t/h_t), and the shocks $\{z_t^g,\, z_t^h\}$. This gives a set of equations, one for each of the data-matched variables, in terms of the state variable and the two shocks. Using US data for the state variable (k_t/h_t), this leaves a set of seven equations in the two unknown shocks $\{z_t^g,\, z_t^h\}$, an overidentification of the shocks. Overidentification allows for a relatively data-invariant backed-out shock, using different combinations of variables, as opposed to an exactly identified shock set from using any two of the data series alone. Then the two backed-out shock series are constructed from the set of variable solutions by following the Benk et al. (2005) method of ordinary least squares. This estimates each shock at each time period t using the seven data points for each time period t, for each shock. More data series is "better" than two because this gives a larger "data sample" from which to estimate each shock at each point in time.

Figure 3's model-based goods TFP shock has a 0.78 correlation with the Solow residual. The high correlation compares to a similar magnitude found in Nolan & Thoenissen (2009), who also back out and compare their TFP model shock to the Solow residual, although using instead a different model that has a financial shock, a money supply growth shock, and a goods sector TFP shock. One very noticeable departure of the two series is that the model TFP sharply falls in 2008 and then gradually begins rising, while the Solow residual continues to fall.

By examining the differences across filtered frequencies, further detail emerges as to where the two depart. Figure 4 shows that in the Business cycle window the Solow residual rises in 2008-9, while the model goods TFP shock moves sharply down, as consistent with the Great Recession. In the low frequency, the model TFP begins rising after 2010 and gradually approaching the baseline (zero) in 2016, while the low frequency Solow residual only rises slightly and remains well below (zero); the same holds in the Medium cycle. This is circumstantial evidence that the model shock is doing a better job of explaining how the US economy cycled sharply down in 2008 and then began a slow gradual recovery, with almost a whole "lost decade" before getting back (to zero here); the model seems to be closer to real GDP experience than the Solow residual.

Figure 5 shows the Model 1 backed out, unfiltered, shock compared to the Solow residual

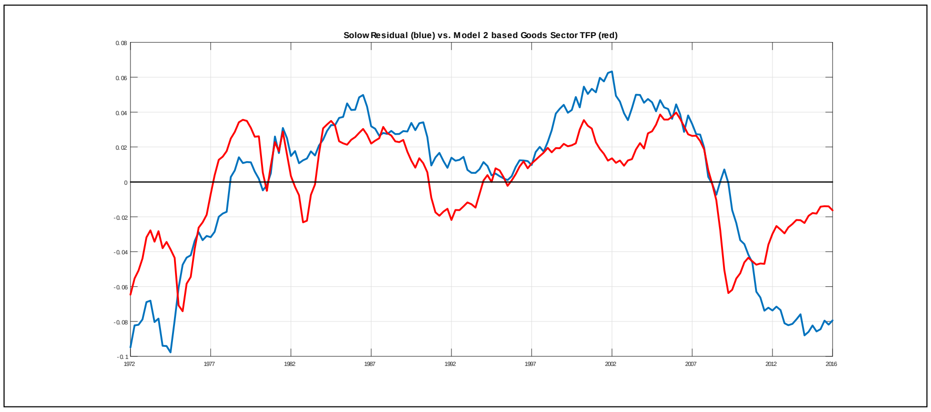

Figure 3: Total Factor Productivity: Model 2 (red) vs Solow residual (blue).

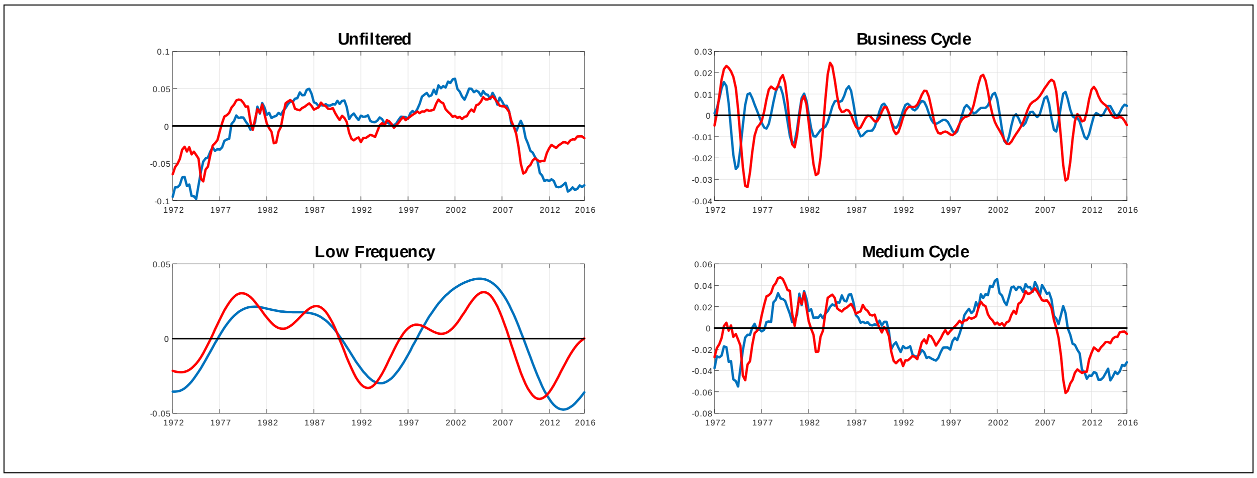

Figure 4: Total Factor Productivity - Model 2 derived (red) versus Solow residual (blue) at different frequencies.

in the upper lefthand tile and the correlation is only 0.27. Tracking the Solow residual better does not prove that Model 2 is better than Model 1, but it provides some circumstantial evidence as to how the models are performing relative to more standard measures of TFP. Model 1 also appears to better capture the post 2010 upturn than the Solow residual. And Figure 6 shows that Model 1 across frequencies, for example, seems to miss a lot of low frequency growth in the 1970's to 1980's, as well as not providing as sharp an upturn post 2010 compared to Model 2.

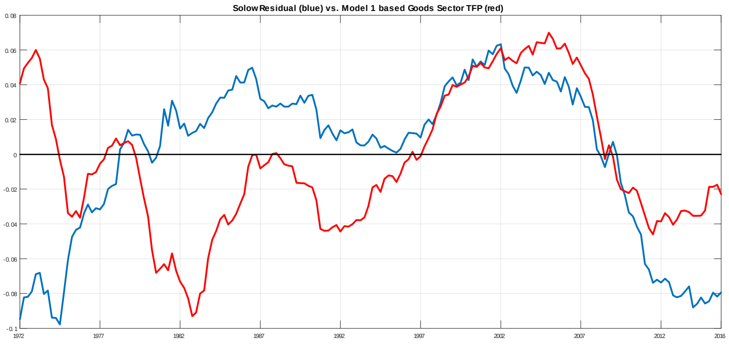

Figure 5: Total Factor Productivty: Model 1 (red) vs Solow residual (blue).

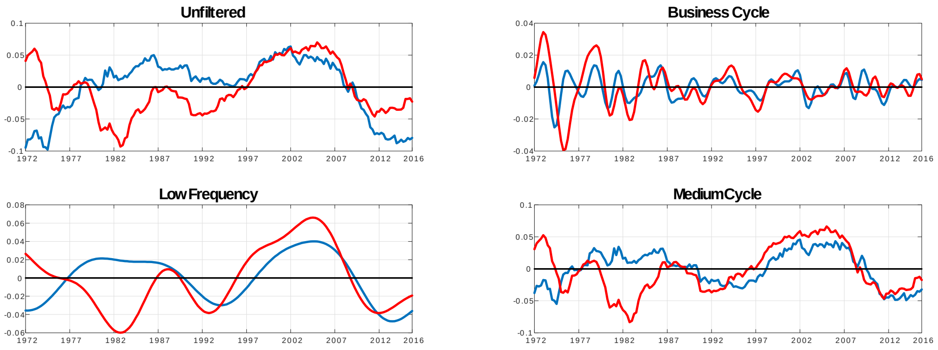

Figure 6: Total Factor Productivity - Model 1 derived (red) versus Solow residual (blue) at different frequencies.

Model 2 appears as the preferred model. For example Feenstra et al. (2015) show a positive rate of increase in TFP post 2010 but one that is well-below its historical trend, as is consistent with Model 2.[11] Also consider how the "Wage Tracker" data shows that the

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[11] "Total Factor Productivity at Constant National Prices for United States"; sourced from Feenstra et al.

median US wage growth fell sharply during the Great Recession and began steadily rising after 2010, very similar to the Figure 3 post-2010 period for Model 2's goods TFP shock.[12]

Finally consider that in Appendix D both the negative and the positive backed-out shock components for the goods sector shock are each graphed separately. In comparison it is clear that the magnitude of the backed out shocks are much smaller for the negative shocks than for the positive shocks. This implies the type of asymmetry that Hansen and Prescott (2005) identify for the US and that Harding and Pagan (2002) identify for both the US and in a broader sample set.

5 Results

Based on a model simulation that follows the methodology of Restrepo-Ochoa & Vazquez (2004), moment results are presented at different frequencies for key correlations, volatilities, and persistence of growth rates.[13] The frequencies are found using the Christiano & Fitzgerald (2003) band-pass filter. The windows are high frequency (HF: 2 - 6 quarters), business cycle frequency (BC: 6 - 32 quarters), low frequency (LF: 32 - 200 quarters), and the Comin & Gertler (2006) 'Medium Cycle' that combines these frequencies (MC: 2 - 200 quarters).

The windows are also used to exhibit the economy's ability to explain the consumption to output ratio that stands at the heart of the many puzzles facing real business cycle theory. Starting with the c/y ratio of the US data, the first results reported are a comparison of the economy's explanation of c/y relative to the data. Added for additional comparison is a construction of c/y using the standard RBC.

5.1 Consumption-Output Ratio

For the quarterly historical US data from 1971:4 to 2015:4, Figure 7 shows the Model 2 constructed c/y for the business cycle (BC) window in green compared to the data for c/y given by the red line.[14] This construction is made from the backed out shocks and the models solution for c/y in terms of k/h and the shocks. The figure shows that the data is tracked rather closely by Model 2, including during the Great Recession and its aftermath.

In contrast, Figure 8 shows in blue the standard RBC model fit of c/y, in the business cycle window, using the backed out TFP shock for the RBC model and the RBC solution of c/y as a function of the shocks and the state variable k. The same actual c/y data is included in red. It is clear that the standard RBC model fits the data less well than does

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(2015); retrieved from FRED, https://fred.stlouisfed.org/series/RTFPNAUSA632NRUG, August 25, 2016.

[12] The Atlanta Fed "Wage Tracker" uses BLS CPS household data, and can act as a proxy of productivity.

[13] An Online Appendix sets how this Restrepo-Ochoa & Vazquez (2004) simulation methodology is equivalent to King et al. (1988).

[14] The data description is given in Appendix B.

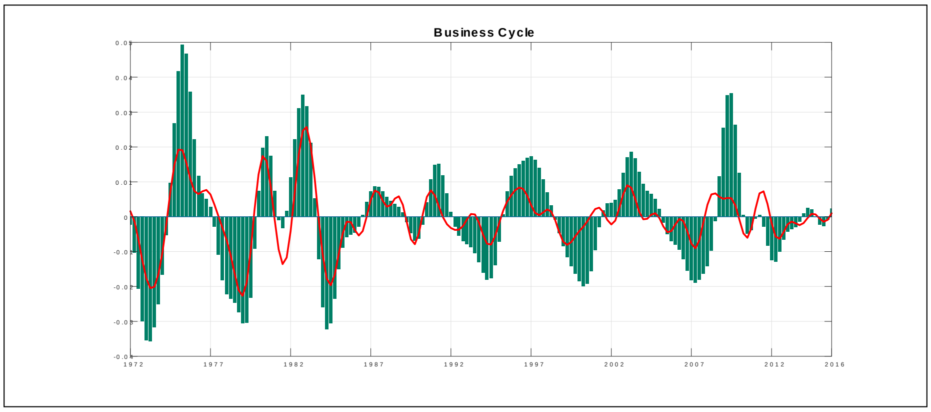

Figure 7: Consumption-Output Ratio (C/Y) at the Business Cycle Frequency: Model 2 total shock contribution (green area) versus US Data (red line).

Model 2 above, in that it have a much less exacting fit of the data, with too much volatility compared to the data and to Model 2.

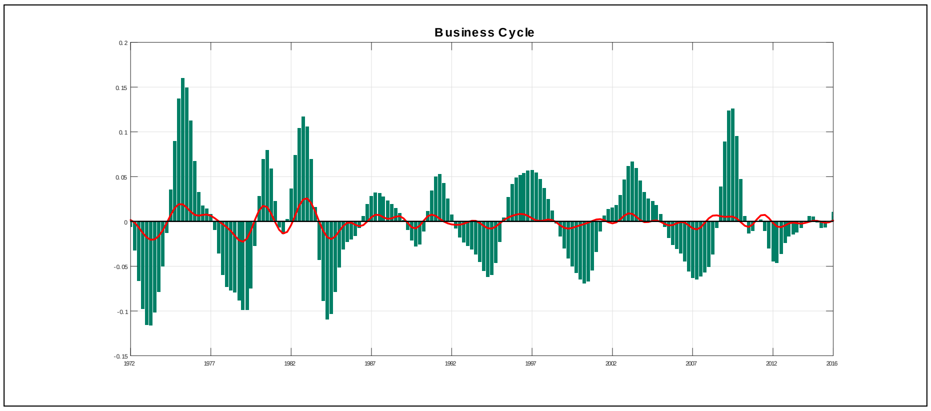

Figure 8: Consumption-Output Ratio (C/Y) at the Business Cycle Frequency: RBC model total shock contribution (green area) versus US Data (red line).

To see a sampling of the low frequency results, Figure 9 shows the Model 2 generated c/y in the Medium Cycle frequency, as compared to the c/y data filtered to the same frequency. Model 2 appears to capture fairly well the actual Medium Term Cycle c/y. To see the comparison to the RBC at the lower frequencies, consider Figure 10. This shows across frequencies the RBC model value for c/y, as given by the black line, compared to the red line for the actual c/y data and to the green shading for the Model 2 c/y. For the unfiltered data, the business cycle, the low frequency and the Medium Cycle, it is clear that the RBC model (black line) is much more volatile compared to the data (red line) and

Model 2 (green).

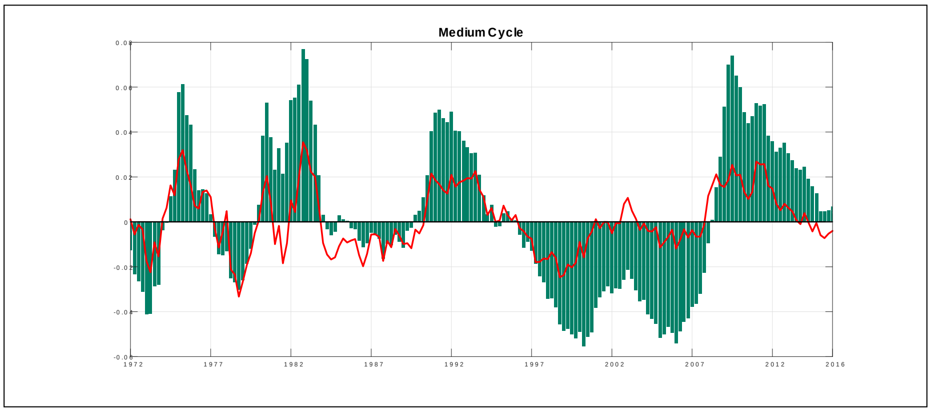

Figure 9: Consumption-Output Ratio (C/Y) at the Medium Cycle Frequency: Model 2 total shock contribution (green area) versus US Data (red line).

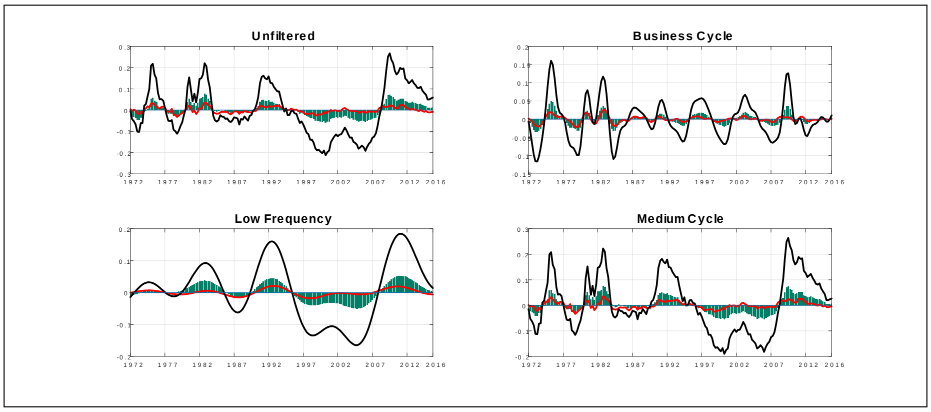

Figure 10: Consumption-Output Ratio (C/Y): Model 2 total shock contribution (green area), RBC model TFP contribution (black line) versus US Data (red line).

5.2 Correlations

Tables 3 - 6 report correlation moments for US data, Model 1 and Model 2. Table 3 shows that the comovement of consumption and investment with output is closely matched by Models 1 and 2 at the business cycle frequency. Both models are able to capture a positive correlation between labor hours and output as suggested by US data at the business cycle, low frequency, and the Comin & Gertler (2006) Medium Cycle, with Model 2 closer to the data. Both models capture the positive business cycle correlation between labor hours and consumption, unlike the standard RBC model. Both models generate a strong negative theoretical correlation between human capital investment time hours and output as suggested in DeJong et al. (1996), and as consistent with certain limited evidence. Model 2 is also able to capture the positive correlation of physical capital utilization rate and output at the business cycle frequency and the lower frequencies, although doing best at the business cycle.

Variable		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(c_t, y_t)$	Data	0.475	0.891	0.980	0.963
	Model 1	0.893	0.776	0.931	0.927
	Model 2	0.989	0.928	0.856	0.837
$corr(i_{kt}, y_t)$	Data	0.809	0.939	0.834	0.833
	Model 1	0.784	0.841	0.691	0.696
	Model 2	0.997	0.991	0.936	0.939
$corr(l_{gt}, y_t)$	Data	0.394	0.732	0.589	0.595
	Model 1	-0.196	0.200	0.027	0.036
	Model 2	-0.141	0.874	0.823	0.819
$corr(l_{ht}, y_t)$	Data	-	-	-	-
	Model 1	0.214	-0.016	0.131	0.111
	Model 2	0.119	-0.891	-0.833	-0.827
$corr(u_t, y_t)$	Data	0.432	0.797	0.447	0.483
	Model 1	-	-	-	-
	Model 2	0.001	0.926	0.871	0.819
$corr(c_t, l_{gt})$	Data	0.206	0.766	0.592	0.596
	Model 1	-0.077	0.672	0.362	0.319
	Model 2	-0.229	0.651	0.378	0.383

Table 3: Matching Correlations (US Data 1959Q1-2015Q4, Model 1 & 2).

Table 4 compares additional correlation moments about shares of output that are less examined in RBC theory, and which were not targeted in the calibration. In particular, following the lead of Hansen and Prescott (2005) in focusing on the labor share of income, as an extension here the paper reports how Model 2 compares in terms of the main component shares of each the Gross Domestic Product (GDP) and Gross Domestic Income (GDI) measures of the National Income and Product Accounts (NIPA), using FRED to source the data. For GDP, accounting for 85% in the fourth quarter of 2015 for example, is the sum of Personal Consumption Expenditure and Gross Private Domestic Investment. In the model, as shares of GDP, these are compared respectively to c/y and i/y. For GDI, accounting for 93% of the data in the same quarter, is the sum of Consumption of Fixed Capital, Net Operating Surplus and Compensation of Employees. In the model, as shares of GDI, these are compared respectively to $\frac{\delta(u_t)k_t}{y_t}$, $\frac{r_t v_{gt} u_t k_t}{y_t}$, and $\frac{w_t(l_{gt}+l_{ht})h_t}{y_t}$. Since Consumption of Fixed Capital draws together the depreciation from both government and private sectors, the education sector's capital depreciation must fall in there as well as so the total capital depreciation $\delta(u_t)k_t$ is used for comparison; for the share of Net Operating Surplus, since the education sector is mainly non-profit only the goods sector share of rental income $r_t v_{gt} u_t k_t$ is used in the ratio; for Compensation, the labor income from both sectors $w_t(l_{gt}+l_{ht})h_t$ is used.

For the GDP shares, Table 4 shows that in both the model and the data c/y is countercyclic while i/y is procyclic, for the BC, LF and MC windows. The magnitudes of these correlations are also close in model and data. For the GDI shares, Table 4 shows that in both model and data the capital depreciation is negative in the BC, LF and MC windows, with a close relation of the magnitudes. The rental income of capital is procyclic in both model and data in the same BC, LF and MC windows, with a close relation of magnitude in the BC window. And finally the labor share of income is negative in both model and data in the BC window, as related to the results of Hansen and Prescott (2005), although of mixed signs in the lower frequency model and data results. Thus the model strikingly has explanatory power for a full set of NIPA shares that comprise what can be considered as key RBC stylized facts about comovements of ratios. In the past, the GDP shares that are well explained here have been known as the Great Ratios, such as in Klein and Kosubud (1961).

Share of GDP, GDI Correlation with y		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(\frac{c_t}{y_t}, y_t)$	Data	-0.83	-0.87	-0.78	-0.75
	Model 2	0.38	-0.69	-0.68	-0.67
$corr(\frac{i_{kt}}{y_t}, y_t)$	Data	0.63	0.87	0.50	0.55
	Model 2	-0.38	0.69	0.68	0.67
$corr(\frac{\delta(u_t)k_t}{y_t}, y_t)$	Data	-0.66	-0.82	-0.52	-0.56
	Model 2	0.35	-0.72	-0.74	-0.68
$corr(\frac{r_t v_{gt} u_t k_t}{y_t}, y_t)$	Data	0.47	0.64	0.04	0.19
	Model 2	-0.32	0.77	0.65	0.68
$corr(\frac{w_t(l_{gt}+l_{ht})h_t}{y_t}, y_t)$	Data	-0.30	-0.29	0.35	0.19
	Model 2	0.23	-0.86	-0.76	-0.73

Table 4: NIPA Moment Tables.

5.3 Volatilities

Tables 5 show that the volatility moments of the data are captured relatively well with Model 1 being better in some cases and Model 2 in others. For example, both models are very close to the data for output growth volatility. The volatility physical capacity utilization rate is matched only in Model 2, albeit improvement here is possible given too little volatility in Model 2 compared to the data.

Variable		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$vol(g_{y,t})$	Data	0.0068	0.0064	0.0038	0.0100
	Model 1	0.0047	0.0037	0.0034	0.0068
	Model 2	0.0050	0.0043	0.0034	0.0074
$vol(g_{c,t})$	Data	0.0038	0.0036	0.0029	0.0059
	Model 1	0.0036	0.0031	0.0069	0.0079
	Model 2	0.0022	0.0017	0.0015	0.0031
$vol(g_{i_k,t})$	Data	0.0200	0.0207	0.0105	0.0302
	Model 1	0.0160	0.0150	0.0260	0.0330
	Model 2	0.0120	0.0110	0.0091	0.0190
$vol(y_t)$	Data	0.0044	0.0166	0.0469	0.0500
	Model 1	0.0034	0.0100	0.0590	0.0600
	Model 2	0.0033	0.0100	0.0360	0.0380
$vol(c_t)$	Data	0.0024	0.0097	0.0382	0.0396
	Model 1	0.0028	0.0068	0.0550	0.0550
	Model 2	0.0015	0.0038	0.0200	0.0200
$vol(i_{kt})$	Data	0.0129	0.0540	0.0912	0.1076
	Model 1	0.0110	0.0420	0.1500	0.1500
	Model 2	0.0081	0.0290	0.0910	0.0960
$vol(l_{gt})$	Data	0.0017	0.0049	0.0221	0.0227
	Model 1	0.0070	0.0112	0.0090	0.0158
	Model 2	0.0037	0.0120	0.0350	0.0370
$vol(u_t)$	Data	0.0055	0.0254	0.0318	0.0420
	Model 1	-	-	-	-
	Model 2	0.0011	0.0023	0.0039	0.0047

Table 5: Countercyclic Data Moments for Labor Share of Output and Input Price Ratio

5.4 Persistence

Table 6 shows model persistence in two ways. First, to follow the literature of Benhabib et al. (2006) and Cogley & Nason (1995), the unfiltered simulated model data is used to generate the autocorrelation profile $\rho(\cdot)$, and compared to the unfiltered actual data. This is done for output growth, consumption growth, physical capital investment growth, and the levels of goods sector labor and the physical capital capacity utilization rate. The main failing of Model 1 is that it only gets the initial level of growth persistence, but not the falling autocorrelation profile as seen in the data. Model 2 better captures both the level and the autocorrelation profile across the four data growth autocorrelations with three lags reported.

With an extension to 16 lags, Figure 11 graphs the three growth autocorrelation profiles, plus the profile for labor, for the data (in blue), for Model 1 (in red) and for Model 2 (in yellow). The four tiles are A: output growth; B: consumption growth; C: physical capital investment growth; and D: goods sector labor. In contrast, as reported by Benhabib et al. (2006), traditional RBC models fail to reproduce the output growth persistence beyond the first lag.

Variable		Lag 1	Lag 2	Lag 3
$\rho(g_{y,t})$	Data	0.270	0.216	0.160
	Model 1	0.636	0.605	0.596
	Model 2	0.271	0.220	0.188
$\rho(g_{c,t})$	Data	0.369	0.284	0.305
	Model 1	0.631	0.610	0.608
	Model 2	0.380	0.361	0.347
$\rho(g_{ik,t})$	Data	0.264	0.177	0.082
	Model 1	0.329	0.265	0.225
	Model 2	0.282	0.213	0.170
$\rho(l_{gt})$	Data	0.987	0.975	0.962
	Model 1	0.956	0.917	0.883
	Model 2	0.993	0.983	0.971
$\rho(u_t)$	Data	0.956	0.863	0.751
	Model 1	-	-	-
	Model 2	0.956	0.919	0.887

Table 6: Simulated Autocorrelation Functions vs. Data (US Data 1959Q1 - 2015Q4).

A second measure of persistence was computed across frequency using filtered data, again with three lags. The results for this are reported in the following section using the metric

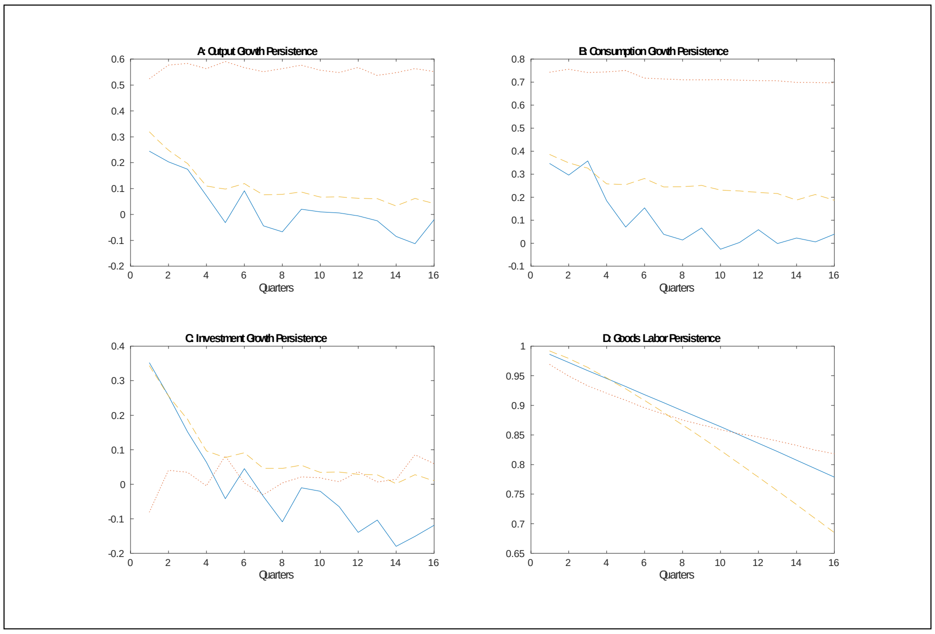

Figure 11: Autocorrelation profiles of variables for 15 quarters: US data-based, 1972Q1-2015Q4, solid blue line; Model 1 simulated data, dotted red line; Model 2 simulated data, dashed yellow line.

measure of the average distance of the model moments from the data moments. This second filtered set of persistence results are denoted by Persistence *, while unfiltered persistence moments are denoted by Persistence **.

6 Metric for Model Comparison

Besides its use in the calibration choice, the other advantage of the distance metric is that it represents the average percentage point deviation of the simulated moments from the US data-based moments. Therefore, it allows further information of model performance for the choice of the calibration, as well as the ability to compare the performances of different DSGE models relative to the data across all moments and/or across subsets of moments.

Table 7A shows the "Overall" average metric across all moments that are reported in Tables 3, 5 and 6 for which there was data for comparison (human capital investment time was excluded). For Model 2, there are 20 Correlation moments, 32 Volatility moments, and 15 unfiltered Persistence ** moments for a total of 67 "Overall"; all of these 67 model moments are targeted in the calibration grid search. The corresponding average metrics respectively are 0.50 for correlations, 0.51 for volatilities, 0.15 for unfiltered persistence, and 0.46 for the overall average. For Model 1, there are 16 Correlation moments, 28 Volatility

TABLE 6A	Average Metric Across All Moments	
	Model 1	Model 2
Overall	0.59	0.46
Correlations	0.50	0.50
Volatilities	0.53	0.51
Persistence*	0.73	0.38
Persistence**	0.95	0.15

TABLE 6B	Average Metric Across Four Frequencies							
	Model 1 HF	Model 2 HF	Model 1 BC	Model 2 BC	Model 1 LF	Model 2 LF	Model 1 MC	Model 2 MC
Correlations	0.95	1.15	0.27	0.15	0.39	0.39	0.40	0.33
Volatilities	0.60	0.51	0.43	0.64	0.70	0.36	0.38	0.51
Persistence*	0.41	0.75	1.91	0.42	0.02	0.08	0.59	0.27

Table 7: Model 1 and 2 percentage deviation based metric for moments.

moments, and 12 (unfiltered) Persistence ** moments for a total of 56 targeted moments (fewer in number than Model 2 because there is not a variable physical capital capacity utilization rate); again all of these reported model moments are targeted in the Model 1 grid search for the calibration. The corresponding average metrics are 0.50 for correlations, 0.53 for volatilities, 0.95 for unfiltered persistence, and 0.59 "Overall".

The filtered persistence (Persistence *) constitutes the remaining row of Tables 7A. This is included (but not targeted or used to calculate the Overall metric) as an alternative measure of persistence to that of using the unfiltered data, as is the focus of the literature. For Model 2, along with the growth rate of output, consumption, investment, and labor is added the capacity utilization growth with 3 lags, so as to give 5x3=15 moments that are averaged within each of the four frequencies, for a total of 60 targets and a resulting average metric of 0.38; dropping the physical capital utilization rate, Model 1 has a total of 48 moments across frequencies with an average metric of 0.73. Results in Table 7A show that Model 2 has a lower average distance metric for the "Overall" calculation, as well as for the correlations and both unfiltered and filtered persistence.

Results in Table 7B show that Model 2 correlations, within the BC, LF and MC windows, have an average 15%, 39% and 33% average deviation, respectively. Model 2 volatilities in the LF window have a 36% average deviation. These Model 2 results are better than Model 1 comparable results. Model 2 also has lower average deviations than Model 1 for the filtered Persistence* in the BC, LF and MC windows, including a very low 8% deviation for Model 2 in the LF window; Model 1 has an even lower 2% deviation for that same window.[15]

For robustness, similar metrics were computed for Model 2 with the assumption of a 1.0 correlation between shocks. The metric tables change little with this exercise. And although a 1.0 correlation did not emerge as the preferred calibration, these results indicate that such a simplification of the shock structure would be plausible.

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[15] Model 1 and 2 were extended with a government sector and corresponding shock, in the fashion of Chari et al. (2007), but did not improve overall on the performance in terms of the distance metric; for example the government model was marginally better in BC volatilities but worse in capturing BC and LF correlations and the autocorrelation profiles of growth rates in Figure 11.

7 Discussion

This paper views the output of each sector as being a function of the Cobb-Douglas combination of the stock of human and the stock of physical capital used within each sector: the flow of output results from these stocks of capital inputs; this is a different interpretation for example than Dejong et al., 1996. While past accumulation determines the current total stock of each capital, during time t the stock of inputs of capital to production can be altered by using either capital in the aggregate more productively or less productively, in the form of a variable leisure that determines human capital usage and a variable u_t that determines physical capital usage. These variable usage rates give strong outlets by which to equalize returns on capital intertemporally as in equation (12), as well as intratemporally balancing the value of unused capital during time t, according to preferences as in equation (11), without altering the stocks of capital that exist at time t. Because preferences include both utilization rates, the extra intratemporal condition provides a second source of capital symmetry, besides equalization of returns, As compared to models without a similar type of intratemporal symmetry in the value of utilized capital, here the ability to alter utilization rates creates a much reduced burden on changing factor prices, reallocating resources between sectors, and on the magnitude of the shock necessary to generate the lower magnitude of the changes in prices and in factor reallocations.

The production of inputs from the outputs of each respective sector sets up a dynamic that reverberates through the Rybczynski (1955) theorem and dually with the Stolper & Samuelson (1941) theorem. The change in sectoral output from a change in an input is equal to the change in the sector's relative price with respect to a change in its input price; this duality is proven in the Appendix C.[16]. What this means is that an economy-wide productivity shock causes simultaneous reallocations where the factor input prices respond to the relative size of the corresponding increase in the factor inputs.

Human capital plays a stealth role here in an accounting sense similar to that played by McGrattan's (2015) intangible capital. Both of these capitals in their respective settings are not accounted for using Solow growth accounting in that the so-called "labor wedge" of Chari et al. (2007) is exactly equal to the time spent in human capital investment here and equal to intangible capital investment in McGrattan (2015). To see this wedge in the notation of Model 2, consider that for an exogenous growth economy as in Chari et al. (2007) a potential wedge τ_{lt} exists between the firm's marginal product of labor and the consumer's marginal rate of substitution between leisure and goods ($MRS_{c,x}$), as defined

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[16] Mulligan & Sala-i Martin (1993), with a linear production function for human capital, and Bond et al. (1996), with a continuous time version of this paper's Model 1, respectively prove a related Stolper & Samuelson (1941) theorem but not its duality to Rybczynski (1955) or its RBC application as shown here.

by :

$$(1-\tau_{lt})\tilde{w}_t = \frac{A\tilde{c}_t}{1-l_{gt}} = MRS_{c,x}, \tag{15}$$

where $\tilde{w}_t = (1-\phi_1)A_g e^{z_t^g}\left[\frac{\tilde{k}_t}{z_t l_{gt}}\right]^{\phi_1} z_t$ and variables with a tilde represent variables normalized by the exogenous growth trend. The wedge equals the share of productive time not used towards goods production.

The Model 2 marginal rate of substitution between goods and leisure can be written as $\tilde{w}_t x_t = A\tilde{c}_t$, or $\tilde{w}_t\left(1-l_{gt}-l_{ht}\right) = A\tilde{c}_t$; here variables with tildes represent h_t -normalized variables. Dividing by the sum of non-market time, $(1-l_{gt})$, yields that $\tilde{w}_t\frac{1-l_{gt}-l_{ht}}{1-l_{gt}} = \tilde{w}_t\left(1-\frac{l_{ht}}{1-l_{gt}}\right) = \frac{A\tilde{c}_t}{1-l_{gt}}$, which gives a related equation to (15) where $\tilde{w}_t\left(1-\tau_{ht}\right) = \frac{A\tilde{c}_t}{1-l_{gt}}$ such that $\tau_{ht} \equiv \frac{l_{ht}}{1-l_{gt}}$.

Yet human capital may explain why capital does not flow freely into less developed economies that lack a sufficient return on human capital (Lucas, 1990); it can explain isomorphically the goods sector TFP as a simple result of human capital accumulation in a Lucas (1988) BGP accounting point of view; and in Lucas (1988) it makes endogenous the Ramsey-Solow growth rate of output. Harding and Pagan (2002, p.380) conclude that "it follows that information upon the evolutionary process for the growth rate in activity needs to be gathered in order to describe the cycle. In particular, the output from theoretical models that is needed relates to the growth rate in output..."

After the focus in Klein & Kosobud (1961), explaining the Great Ratios such as c/y and i/y as a by-product of using an "evolutionary process" for growth in terms of human capital investment, shows robustness of the paper's approach that allows for a relatively small variance of its productivity shocks, despite the Great Ratios not having been targeted in the calibration. As explaining c/y and i/y goes to the heart of explaining the consumption-output comovement, it makes sense that a RBC model might focus on explaining Great Ratios as well. Targeting some moments of the output growth rate during the calibration might also be useful.

8 Conclusion

The paper's dramatic shock amplification results because the economy-wide temporary shock creates a permanent income effect that raises consumption, output and the capital stocks permanently. This is a result of shocking the investment rate of human capital as well the goods sector TFP with an above 99% correlation that defines what is called the economy-wide shock. The magnitude of the variance of the Model 2 economy-wide shock is some 7500 times smaller than the traditional RBC TFP shock variance, and the human capital shock variance 22000 times smaller.

The paper shows that the model can improve traditional RBC data moment matching of correlations, volatilities and output growth persistence. In addition, physical capital utilization rate procyclic moments and human capital time's countercyclical moments are captured as is the level and autocorrelation profile of the growth persistence of output, consumption and investment, along with that profile for labor. Model 2 also captures the countercyclic labor share of output, the countercyclic capital depreciation as a share of output, and business cycle asymmetry as evidenced by the backed-out goods sector TFP shock. This implies that Model 2 provides a simultaneous tuning into both growth, or low frequency, spectra and the business cycle, while reproducing well the historical data on the consumption to output ratio during the business cycle and lower frequencies.

Key to producing the model results is a calibration that employs a deep grid search while demanding both state variable convergence (Blanchard & Quah 1989) and iterative convergence of the model's shock properties to those properties of the backed-out shocks (Benk et al. 2005). It results from historically backing out the model's goods sector productivity shock from US data that the backed out goods TFP productivity shock rises sharply post 2010 albeit at a below trend rate , similar to Feenstra et al. (2015), but unlike the traditionally constructed TFP shock which rises much less post 2010. This may be indicating that the model is better capturing the post 2010 recovery through its human capital channel which is hard to account for in the standard RBC accounting that lies behind the Solow residual.

The metric used in the grid search is the sum of all of the fractional moment deviations of model from data, as normalized by dividing by the total number of moment targets. This makes the aggregate metric the average fractional deviation of the model's moments from the data moments across four spectra of a large moment set. The use of the metric resulted in the leap in the amplification of the shock and the other salient results. The grid search enables a comprehensive calibration space search with the aggregate metric providing a measure of the model's performance as well as a tool to limit the focus of the parameter space to that in which the lowest fractional deviations of the metric were found. Although they were not targets in this paper's calibration, the success in model matching of the accounting shares of GDP and GDI suggest these as useful targets of RBC research.

Future research could include estimating confidence intervals for the calibration methodology by building upon the Simulated Method of Moments (SMM) literature, and could comprise a full comparison to Bayesian methods. Better explaining the volatility of the capacity utilization rate of physical capital and the magnitude of the equity premium with the human capital model (Li 2000) are left for future research. Further grid search work could eliminate the high frequency targets in order to sharpen the focus on business cycle and low frequencies, or alternatively, certain desired targets could be selectively weighted, which is not done here. Heterogeneous agent financial frictions might be added following Buera

& Moll (2016); heterogeneous agents with different human capital productivity may hold promise given the De Giorgi & Gambetti (2017) results of how highly educated individuals in the tail end of the distribution can play an important role in driving cyclic change. Although without human capital, (Oberfield & Raval 2014) finds an elasticity of substitution in production between labor and capital at around 0.8, less than the 1.0 of Cobb-Douglas, which suggests it may be worthwhile to extend the production function of each sector to a more general contant elasticity of substitution one.

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Appendices

A Equilibrium Conditions

Define the Lagrange multiplier of the social resource constraint as λ_t, and that of the human capital accumulation's as χ_t. Then the social planner's first order conditions are the following,

$$c_t: \quad c_t^{-\sigma} x_t^{A(1-\sigma)} (1-u_t)^{B(1-\sigma)} = \lambda_t; \tag{16}$$

$$l_{gt}: \quad A c_t^{1-\sigma} x_t^{A(1-\sigma)-1} (1-u_t)^{B(1-\sigma)} = \lambda_t w_t h_t; \tag{17}$$

$$l_{ht}: \quad A c_t^{1-\sigma} x_t^{A(1-\sigma)-1} (1-u_t)^{B(1-\sigma)} = \chi_t (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} h_t; \tag{18}$$

$$\begin{aligned} u_t: \quad & B c_t^{1-\sigma} x_t^{A(1-\sigma)} (1-u_t)^{B(1-\sigma)} = \lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (v_{gt} k_t) + \\ & + \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (v_{ht} k_t) - \lambda_t \delta_k u_t^{\psi - 1} k_t; \end{aligned} \tag{19}$$

$$v_{gt}: \quad \lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (u_t k_t) = \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (u_t k_t); \tag{20}$$

$$\begin{aligned} k_{t+1}: \quad \lambda_t = {} & \beta E_t \lambda_{t+1} \left[1 + r_{t+1} u_{t+1} v_{gt+1} - \frac{\delta_k}{\psi} u_{t+1}^{\psi}\right] + \\ & + \beta E_t \chi_{t+1} \phi_2 A_h e^{z_{t+1}^h} \left[\frac{v_{ht+1} u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}}\right]^{\phi_2 - 1} (u_{t+1} v_{ht+1}); \end{aligned} \tag{21}$$

$$h_{t+1}: \quad \chi_t = \beta E_t \chi_{t+1} \left[1 + (1-\phi_2) A_h e^{z_{t+1}^h} \left[\frac{v_{ht+1} u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}}\right]^{\phi_2} l_{ht+1} - \delta_h\right] + \beta E_t \lambda_{t+1} w_{t+1} l_{gt+1}; \tag{22}$$

$$y_t = A_g e^{z_t^g}(v_{gt}u_tk_t)^{\phi_1}(l_{gt}h_t)^{1-\phi_1}; \tag{23}$$

$$A_g e^{z_t^g}(v_{gt}u_tk_t)^{\phi_1}(l_{gt}h_t)^{1-\phi_1} = c_t + k_{t+1} - k_t + \frac{\delta_k}{\psi}u_t^{\psi}k_t; \tag{24}$$

$$A_h e^{z_t^h}(v_{ht}u_tk_t)^{\phi_2}(l_{ht}h_t)^{1-\phi_2} = h_{t+1} - (1-\delta_h)h_t; \tag{25}$$

$$1 = v_{ht} + v_{gt};$$

$$x_t = 1 - l_{gt} - l_{ht}. \tag{26}$$

where r_t and w_t denote the marginal productivity conditions of physical and human capital whereby $r_t \equiv \phi_1 A_g e^{z_t^g}(v_{gt}u_tk_t)^{\phi_1-1}(l_{gt}h_t)^{1-\phi_1}$ and $w_t \equiv (1-\phi_1)A_g e^{z_t^h}(v_{gt}u_tk_t)^{\phi_1}(l_{gt}h_t)^{-\phi_1}$. The set of 12 equations (16) - (26) and the two factor marginal product conditions fully describe Model 2, with the 12 unknowns $\{y_t, k_t, h_t, c_t, u_t, l_{gt}, l_{ht}, x_t, v_{gt}, v_{ht}, \lambda_{t,}\chi_t\}$. [17] On the balanced growth path equilibrium, the conditions reduce to a system of two nonlinear equations in two variables, such as g and u, which can be solved numerically for the baseline calibration of parameters defined in Table 1.

B Data Description

The US data used in this paper is from 1959:Q1 until 2015:Q4 except for that of the physical capital utilization rate, which is only available from 1971:Q4, and human and physical capital data, which is available only until the end of 2012. In constructing real data series for US macroeconomic variables Gomme & Rupert (2007) have been followed. Analogously to their methodology the aggregate series are constructed as:[18]

1. *Nominal Market Investment* = Non-residential Fixed Investment + Change in Private Inventories

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[17] Model 1 equilibrium conditions are identical except that $u_t = 1$, $\delta(u_t) = \delta_k$, and $(1-u_t)^{B(1-\sigma)} = 1$.

[18] The raw series and the construction of the underlying data series can be found in *data.xls* included with the Matlab files upon request.

2. *Nominal Home Investment* = Residential Fixed Investment + PCE on Durables
3. *Nominal Investment* = Nominal Home Investment + Nominal Market Investment
4. *Real Investment*= Nominal Investment / (Average Price Deflator / 100)
5. *Nominal Market Output* = Gross Domestic Product - PCE: Housing Services
6. *Nominal Private Market Output* = Nominal Market Output - Employee Compensation: Government
7. *Real Market Output* = Nominal Market Output / (Average Price Deflator / 100)
8. *Real Private Market Output* = Nominal Private Market Output / (Average Price Deflator / 100)
9. *Physical Capital Utilization Rate* = Total Capacity Utilization: Manufacturing
10. *Labor Hours* = Non-farm Business Sector: Average Weekly Hours
11. *Nominal Market Consumption* = PCE on Non-durable Goods + PCE on Services - PCE on Housing Services
12. *Real Market Consumption*= Nominal Market Consumption / (Average Price Deflator/100)
13. *Average Price Deflator* = (Implicit Price Deflator:Non-durables + Implicit Price Deflator: Services)/2

According to Gomme & Rupert (2007), output (y) is measured by real per capita GDP less real per capita Gross Housing Product as defined above. It is due to the argument that home sector production should be removed when calculating market output using the *National Income and Product Accounts* (NIPA). The price deflator is constructed by taking the average of the implicit price deflators on non-durables and services. Population is measured by the number of non-institutionalized persons aged over 16 years. Consumption (c) is measured by real personal expenditures on non-durables and services less Gross Housing Services. Investment is measured by the sum of real Non-residential Fixed Investment, the Change in Private Inventories, Residential Fixed Investment, and Personal Consumption Expenditures on durables. Lastly, working hours are measured by the average weekly labor hours.

The annual index of human capital per person data series is based on years of schooling [Barro & Lee (2013)], and returns to education [Psacharopoulos (1994)]. The series have been constructed by Feenstra et al. (2013) using the perpetual inventory method. Quarterly human capital data has been interpolated using the annual data of Feenstra et al. (2013)

by following Baier et al. (2004) where they define the depreciation rate to human capital as the average of death rates in different age groups for which the data has been obtained from the Center for Disease Control (CDC) database. Also, for the period after 2012 the human capital data has been forecasted by fitting it to an AR1 process. The quarterly physical capital data is constructed from Bureau of Economic Analysis (BEA) annual US capital stock estimates and quarterly data on investment expenditures.

Description	Units	Seasonally Adjusted	BEA / BLS Code / Source	Frequency	Time Range
PCE on Nondurable Goods	Billions of Dollars	SAAR	DNDGRC1	Quarterly	1959Q1 - 2015Q4
PCE on Services	Billions of Dollars	SAAR	DSERRC1	Quarterly	1959Q1 - 2015Q4
PCE on Housing Services	Billions of Dollars	SAAR	DHUTRC1	Quarterly	1959Q1 - 2015Q4
Implicit Price Deflator (Nondurables)	Index 2009=100	SA	DNDGRD3	Quarterly	1959Q1 - 2015Q4
Implicit Price Deflator (Services)	Index 2009=100	SA	DSERRD3	Quarterly	1959Q1 - 2015Q4
Gross Domestic Product	Billions of Dollars	SAAR	A191RC1	Quarterly	1959Q1 - 2015Q4
Employee Compensation: Government	Billions of Dollars	SAAR	B202RC1	Quarterly	1959Q1 - 2015Q4
Civ. Noninst. Pop. 16 and over	1000s of Persons	NSA	CNP160V	Quarterly	1959Q1 - 2015Q4
Capacity Utilization: Manufacturing	Percentage	SA	TCU:MAN.	Quarterly	1971Q4 - 2015Q4
Nonfarm Bus. Sector: Avg. Weekly Hours	Index 2009=100	SA	PRS585006023	Quarterly	1959Q1 - 2015Q4
Nonres. Fixed Investment	Billions of Dollars	SAAR	A008RC1	Quarterly	1959Q1 - 2015Q4
Residential Fixed Investment	Billions of Dollars	SAAR	A011RC1	Quarterly	1959Q1 - 2015Q4
Change in Private Inventories	Billions of Dollars	SAAR	A014RC1	Quarterly	1959Q1 - 2015Q4
PCE on Durables	Billions of Dollars	SAAR	DDURRC1	Quarterly	1959Q1 - 2015Q4
Index of Human Capital per Person	Index	NSA	Penn World Tables 8.0	Annual	1950 - 2012

Table 8: Raw Data Sources.

C Duality Theorems

Stolper & Samuelson (1941) and Rybczynski (1955) theorems underlie the movement of resources between sectors of the general Model 2. Duality of the theorems results because as the input factors increase, they cause a change in the relative price of sectoral outputs; because the outputs of the sectors are physical capital and human capital, changes in the outputs in turn determine changes in input prices.

Proposition 4 *Rybczynski (1955) effect: In the two-sector economy of Section 3, an increase in the allocation of a factor input to a sector will expand the output of that sector if it is more intensive in the increased input; the output of the other sector more intensive in the other factor input will decrease or increase by a relatively lower quantity.*

Proof. One sector produces goods y_t (or alternatively physical capital investment) and the other sector produces human capital investment i_{ht} at a relative price $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$ in terms of the goods output. From equation 23 of Appendix A, the relative price is the ratio of the marginal products with respect to human capital of each the goods and human capital investment investment sectors: $p_{ht} = \frac{(1-\phi_1)A_g z_t^g \left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1}}{(1-\phi_2)A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2}}$. Let the change in human capital investment with respect $v_{ht}u_t k_t$ and $l_{ht}h_t$ be denoted by R_1^h and R_2^h respectively, where $R_1^h \equiv \frac{\partial i_{ht}}{\partial v_{ht}u_t k_t}$, and $R_2^h \equiv \frac{\partial i_{ht}}{\partial l_{ht}h_t}$. It follows that $R_1^h = \phi_2 A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2 - 1} = \frac{r_t}{p_{ht}}$, and that $R_2^h = (1-\phi_2)A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2} = \frac{w_t}{p_{ht}}$. Given that $(1-\phi_2) > \phi_2$, it results that $R_1^h < R_2^h$. ■

Increasing human capital by a unit will increase output of the human capital investment by more than would increasing physical capital by a unit; conversely for the goods sector.

Proposition 5 *Denote by S_1^h and S_2^h the change in the real interest rate and in the wage rate with respect to a change in the relative price of human capital, such that $S_1^h \equiv \frac{\partial r_t}{\partial p_{ht}}$ and $S_2^h \equiv \frac{\partial w_t}{\partial p_{ht}}$. It results that $S_1^h = \frac{r_t}{p_{ht}}$ and $S_2^h = \frac{w_t}{p_{ht}}$.*

Proof. From equation (20) in Appendix A and the definitions for $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$ and r_t, it follows that $\phi_1 A_g e^{z_t^g} \left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1 - 1} = \left(\frac{\chi_t}{\lambda_t}\right) \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt})u_t k_t}{l_{ht}h_t}\right]^{\phi_2 - 1}$, and so that $r_t = p_{ht}\phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt})u_t k_t}{l_{ht}h_t}\right]^{\phi_2 - 1}$. Therefore $S_1^h = \frac{\partial r_t}{\partial p_{ht}} = \phi_2 A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h}\right]^{\phi_2 - 1} = \frac{r_t}{p_{ht}}$. Similarly, equations (17) and (18) in Appendix A imply $S_2^h = \frac{\partial w_t}{\partial p_{ht}} = (1-\phi_2)A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2} = \frac{w_t}{p_{ht}}$. ■

Corollary 6 *Duality between Stolper-Sameulson and Rybczynski effects: The change in the output of the human capital investment sector with respect to a change in an input is equal respectively to the change in that input's implicit competitive price with respect to a change in the implicit relative price of human capital investment to goods output.*

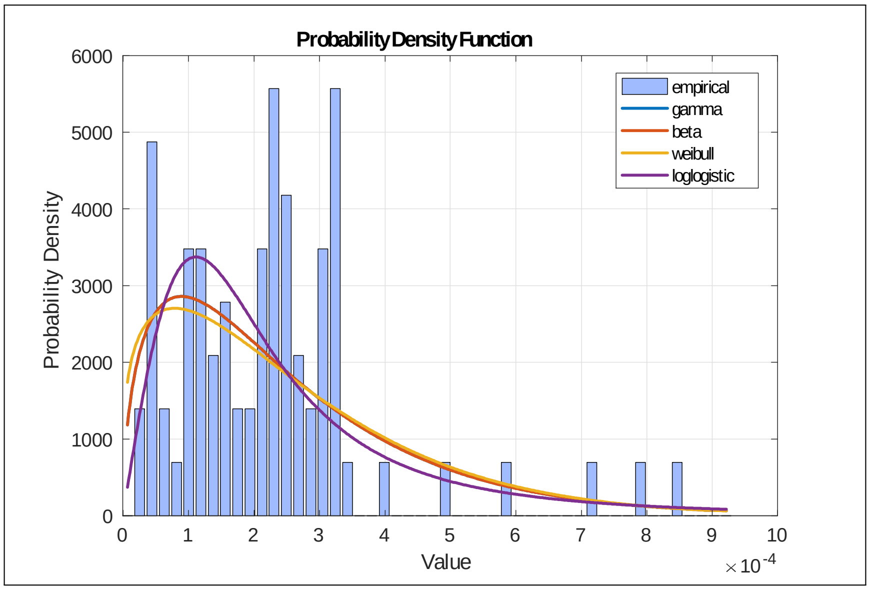

Figure 12: Positive "Backed-out" Goods Sector TFP Shocks: Magnitude of Probability Density and Estimated Distributions.

Proof. Propositions 4 and 5 imply that $R_1^h = S_1^h$ and $R_2^h = S_2^h$. ■

Conversely, duality can be shown for the goods sector by using identical steps if the relative price of goods (physical capital) is defined inversely to p_{ht} as $\frac{\lambda_t}{\chi_t}$. Then it follows that with $S_1^g \equiv \frac{\partial\left(\frac{\partial i_{ht}}{\partial v_{ht} u_t k_t}\right)}{\partial\left(\frac{\lambda_t}{\chi_t}\right)}$ and $S_2^g \equiv \frac{\partial\left(\frac{\partial i_{ht}}{\partial l_{ht} h_t}\right)}{\partial\left(\frac{\lambda_t}{\chi_t}\right)}$, and $R_1^g \equiv \frac{\partial y_t}{\partial v_{gt} u_t k_t}$ and $R_2^g \equiv \frac{\partial y_t}{\partial l_{gt} h_t}$, that $R_1^g = S_1^g = r_t$ and $R_2^g = S_2^g = w_t$.

D Backed-Out Shock Positive and Negative Properties

Figure 12 show the positive range of the backed-out goods sector TFP shock, as compared to the negative range in Figure 13, with different distributions fit to the data as listed in the graph legends. Figure 12 has an average median probability density around 3000 while Figure 13 has one around 1000. Positive values go in the direction of "Trough to the Peak", using Harding and Pagan (2002) terminology, while negative values that go in the "Peak to Trough" direction. These positive shocks have a substantially higher magnitude on average than the negative shocks, consistent with higher, or stronger, positive upswings in the economy and less strong, negative, downturns in the economy, in terms of the model's backed-out TFP shock in Section 4.3. This presents an asymmetry consistent with that found in US data by Harding and Pagan (2002) and Hansen and Prescott (2005).

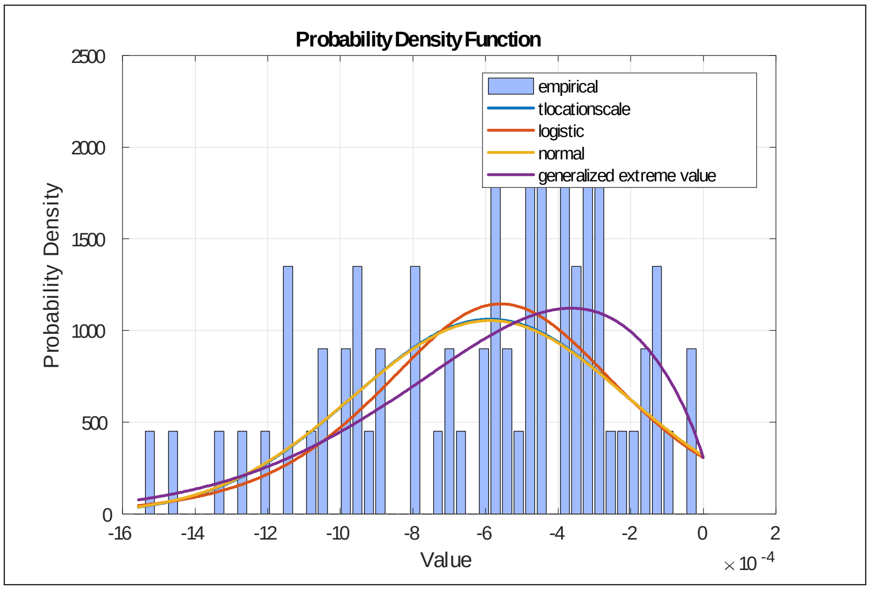

Figure 13: Negative "Backed-out" Goods Sector TFP Shocks: Magnitude of Probability Density and Estimated Distributions.