Factor-Specific Technology Choice*

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February 13, 2017

Abstract

We analyze the properties of a two-dimensional problem of factor-specific technology choice subject to a technology menu – understood as the choice of the degree of factor augmentation by a producing firm or the choice of quality of goods demanded by a consumer. By considering the problem in its generality, we are able to reach beyond the known results for Cobb–Douglas, CES, Leontief (minimum) and maximum functions. We demonstrate that the technology menu and the aggregate function (envelope of local functions) are dual objects, in a well-defined generalized sense of duality. In the optimum, partial elasticities of (i) the local function, (ii) the technology menu and (iii) the aggregate function are all equal and there exists a clear-cut, economically interpretable relationship between their curvatures. Invoking Bergson's theorem, we also comment on the consequences of assuming homotheticity of the three objects, with a particular focus on technology menus constructed as level curves of idea (unit factor productivity) distributions.

Keywords: technology choice, technology menu, production function, utility function, duality, envelope, homotheticity.

JEL codes: C62, D11, D21, E21, E23, O47.

1 Introduction

The purpose of this article is to provide a detailed treatment of a static, two-dimensional problem of optimal factor-specific technology choice. In such a problem, the decision maker faces a menu of local technologies which depend on the quantity of the two factors and their respective quality (i.e., unit productivity). The menu features a trade-off insofar as choosing higher quality of one factor comes at the cost of reducing the quality of the other one. The decision maker is allowed to select her preferred technology, in order to maximize total output/profit/utility, for all configurations of factor quantities. The aggregate function is then constructed as an envelope of local functions, as in Figure 1.

Decision problems with this structure may arise in firms which contemplate not just about the demand for production factors – such as capital and labor – but also about the degree of their technological augmentation (see e.g. Atkinson and Stiglitz, 1969; Basu and Weil, 1998; Caselli

*First draft: January 2017. The author thanks Miguel León-Ledesma, Ingmar Schumacher and Joanna Tyrowicz for their helpful comments and suggestions. The views presented in this article belong to the author and have not been endorsed by the NBP. All remaining errors are entirely the author's fault.

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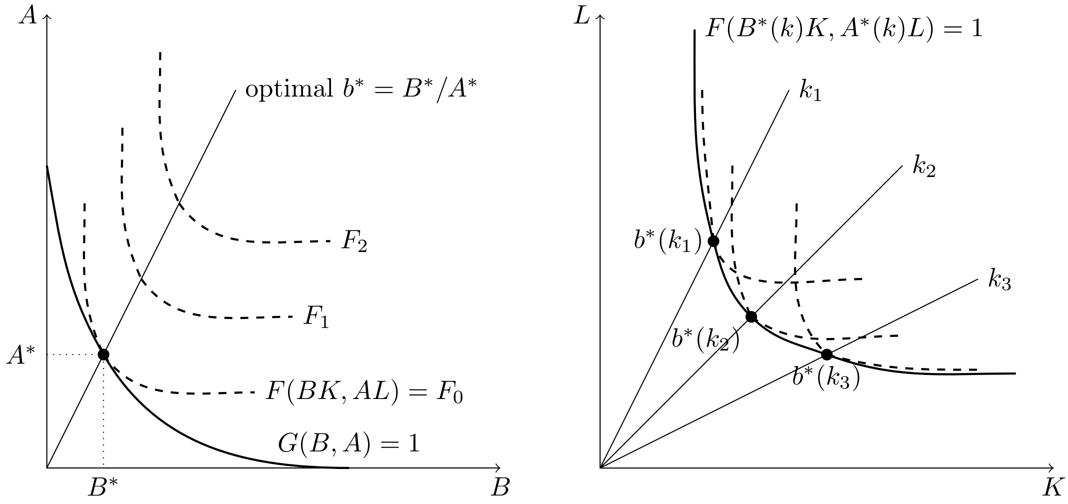
Figure 1: Construction of the aggregate function from local functions by incorporating the optimal factor-specific technology choices.



Step 1. Optimal choice of factor-specific technologies B and A for given quantities K and L

Step 2. Construction of the aggregate function $\Phi(K, L)$ as an envelope of local functions $F(BK, AL)$

and Coleman, 2006). Mathematically equivalent problems are also faced by consumers who are allowed to decide both about the quantity and quality of the demanded goods, as well as by workers (or managers) who allocate their limited endowments of time/effort across two alternative tasks. Hence, despite being motivated primarily by the earlier contributions to the theory of economic growth and factor-augmenting technical change (e.g., Basu and Weil, 1998; Acemoglu, 2003; Jones, 2005; Caselli and Coleman, 2006, as well as the closely related studies listed below), the appeal of the current paper is in fact much broader. The class of problems which we solve here has applications both in micro- and macroeconomics, and they can be viewed both as producer and consumer problems. Factor-specific technology choice problems of the type studied here are useful, in particular, for addressing issues related to natural resources,[1] human capital and capital–skill complementarity,[2] industrial organization, international trade, labor markets, sectoral change, consumption patterns, social welfare, and so on.

Interesting results have already been obtained for certain specific cases of the factor-specific technology choice problem. First, it has been demonstrated that when the technology menu has the Cobb–Douglas form (which may arise, among other cases, if factor-specific ideas are independently Pareto–distributed) (Jones, 2005) or if the local function is of such form (Growiec, 2008a), then the aggregate function also inherits the Cobb–Douglas form. Second, combining a local function of a CES or a minimum (Leontief) form and a CES technology menu yields an aggregate CES function (Growiec, 2008b; Matveenko, 2010; Growiec, 2013; León-Ledesma and Satchi, 2016).[3] Third, a broader treatment of the properties of factor-specific technology choice problems with a

[1]Factor-specific technology choice problems arise naturally when studying the substitutability between exhaustible resources and accumulable physical capital (or renewable resources, cf. Dasgupta and Heal, 1979; Bretschger and Smulders, 2012) as well as human capital (or quality-adjusted labor, cf. Smulders and de Nooij, 2003).

[2]The choice of degree of factor augmentation becomes an important issue once one acknowledges that skilled and unskilled labor are imperfectly substitutable (e.g., Caselli and Coleman, 2006) and potentially complementary to capital (Krusell, Ohanian, Ríos-Rull, and Violante, 2000; Duffy, Papageorgiou, and Perez-Sebastian, 2004).

[3]The implications of factor-specific technology choice in the CES case have been also studied by Nakamura and Nakamura (2008); Nakamura (2009).

minimum (Leontief) local function, including their intriguing duality properties, has been provided by Rubinov and Glover (1998); Matveenko (1997, 2010); Matveenko and Matveenko (2015).[4] While instructive, the minimum function is however an extreme case, particularly problematic when interpreted as a utility function. Fourth, a few promising results for the general factor-specific technology choice problem with an implicitly specified technology menu have also been provided in section 2.3 of León-Ledesma and Satchi (2016).

Notwithstanding these important special cases, the literature thus far has not devised a general theoretical framework allowing to analyze the factor-specific technology choice problem in its generality. The key contribution of this article is to put forward such a general theory – one which would frame all these earlier results in a unique encompassing structure. We find that a unique optimal factor-specific technology choice exists for any homothetic local function F and technology menu G. Plugging this choice into the local function F leads to a unique homogeneous (constant returns to scale) aggregate function Φ, which may then be transformed to a homothetic form by an arbitrary monotone transformation. We also find that (i) the shape of the aggregate function Φ depends non-trivially both on F and G unless one of them is of the Cobb–Douglas form, and (ii) the aggregate function Φ offers more substitution possibilities (i.e., has less curvature) than the local function F unless the optimal technology choice is independent of factor endowments, which happens only if F is Cobb–Douglas or G follows a maximum function.

Our second contribution is to construct and solve the dual problem (in a well-defined generalized sense of duality) where, for every technology, the decision maker maximizes output/profit/utility subject to a requirement of producing a predefined quantity with the aggregate technology Φ. Then, by plugging these optimal factor choices into the local function F, we obtain the technology menu G as an envelope. The results are fully analogous.[5]

At this stage, the duality property also allows us to provide an additional contribution. Namely, we find that in the optimum, partial elasticities of all three objects – the local function F, the technology menu G and the aggregate function Φ – are all equal. We then identify a clear-cut, economically interpretable relationship between their curvatures, giving rise to interesting qualitative implications on concavity/convexity and gross complementarity/substitutability along the three functions (see section 4).[6]

The assumption of homotheticity which we make throughout the analysis, while shared by bulk of the associated literature, does not come without costs. The key limitation is due to Bergson's theorem (Burk, 1936) which states that every homothetic function that is also additively separable (either directly or after a monotone transformation) must be either of the Cobb–Douglas or CES functional form. Hence, when one wants to go beyond the CES framework, one must either give up homotheticity (e.g., Zhelobodko, Kokovin, Parenti, and Thisse, 2012) or additive separability (e.g., Revankar, 1971; Growiec and Mućk, 2016, this paper). It follows that all the non-CES cases which

[4] See also the book by Rubinov (2000).

[5] That is, a unique optimal technology-specific factor choice exists for any homothetic local function F and aggregate function Φ. Plugging this choice into the local function F leads to a unique homogeneous technology menu G, which may then be transformed to a homothetic form by an arbitrary monotone transformation. The shape of the technology menu G depends non-trivially both on F and Φ unless one of them is of the Cobb–Douglas form. Finally, the technology menu G offers more substitution possibilities (i.e., has less curvature) than the local function F unless the optimal factor choice is independent of technology, which happens only if F is Cobb–Douglas or Φ follows a maximum function.

[6] Our findings also underscore that the case with a Cobb–Douglas technology menu and a Cobb–Douglas aggregate function, studied in detail by Jones (2005) and León-Ledesma and Satchi (2016), is in fact very special and not well suited for drawing general conclusions.

are covered by the current study but have not been discussed before, cannot be written down as additively separable.

We also devote a separate section of the paper to study the link between the technology menu and the distributions of ideas. Indeed, part of the associated literature derives the technology menu as a level curve of a certain joint distribution of ideas (unit factor productivities) where the marginal idea distributions are either independent (Jones, 2005; Growiec, 2008b) or dependent following a certain copula (Growiec, 2008a). Extending these studies, we show that such "probabilistic" construction of the technology menu may place a restriction on the considered class of functions G, potentially reducing it to the Cobb–Douglas or CES form because of their homotheticity and additive separability (after a monotone transformation). To show this, we adapt Bergson's theorem (Burk, 1936) to the case of copulas, and particularly Archimedean ones.

The paper is structured as follows. Section 2 presents the setup of the considered problem. In section 3 we derive the optimal technology choice. In section 4 we plug it into the local production function and thus build the envelope. Section 5 discusses the most instructive special cases known from the literature. Section 6 studies the link between the technology menu and distributions of ideas. Section 7 concludes.

2 The Primal and Dual Optimization Problem

2.1 The Primal Problem

In the primal problem, the decision maker (the output- or profit-maximizing firm, the utility-maximizing consumer) maximizes a local function $F(BK, AL)$ with respect to the technology pair (B, A) taken from a level curve of the technology menu $G(B, A)$, taking $K > 0$ and $L > 0$ as given.[7] The aggregate function $\Phi(K, L)$ is obtained as an envelope, by plugging the optimal choices $(B^*(K, L), A^*(K, L))$ into the local function. Formally, we write:

$$\Phi(K,L) = \max_{(B,A)\in\Omega_G} F(BK, AL) \qquad s.t. \qquad \Omega_G = \{(B,A) \in \mathbb{R}^2_+ : G(B,A) = 1\}. \tag{1}$$

In the basic treatment of the static problem (1), it is assumed that the local function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous (constant-returns-to-scale, CRS). Homogeneity permits to rewrite F in its intensive form, $F(BK, AL) = F\left(\frac{BK}{AL}, 1\right) AL = f(bk)AL$, where $b = B/A$ and $k = K/L$. The local function F is interpreted as the local (short-run, exogenous-technology) production function faced by a firm or utility function of a consumer. Each of its arguments is a product of a quantity (K or L) and its quality multiplier, i.e., unit factor productivity (B or A, respectively). Finally, while mathematically this is not necessary, economic interpretation of the local function implies that in typical applications, it should be concave in each of its arguments.

Symmetrically, we also assume that the technology menu $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous. Analogously, we rewrite G in its intensive form,

[7] We denote the quantities K and L so that they are easily recognized as "capital" and "labor", in line with the production function interpretation of the discussed framework. However, this is done only to keep the discussion close to the associated literature. In fact, the theory can be applied just as well to utility maximization problems, where K and L are understood as quantities of two goods demanded by a consumer, as well as to production functions with any other pair of inputs.

$G(B, A) = G\left(\frac{B}{A}, 1\right) A = g(b)A$. The technology menu G is a function which maps factor-specific quality levels to a scalar, interpreted as an overall "technology level" of the economy as faced by the decision maker. Under the production function interpretation, we say that the larger is the value of G, the more can be produced from given inputs; under the utility function function interpretation, the value of G scales total utility attainable from the given endowment of goods.

In a slight generalization of the above problem, we replace homogeneous functions F and G with their homothetic counterparts, respectively $F_h = f_h \circ F$ and $G_h = g_h \circ G$, where $f_h, g_h : \mathbb{R}_+ \to \mathbb{R}$ are monotone (typically, increasing) and twice continuously differentiable. This additional degree of freedom is particularly useful in the analysis of utility which is often viewed as an ordinal rather than cardinal concept.

2.2 The Dual Problem

In the dual problem, the decision maker maximizes a local function $F(BK, AL)$ with respect to the quantities (K, L), subject to maintaining a predefined level of output/utility given by the aggregate function $\Phi(K, L)$, and taking the factor-specific technologies $B > 0$ and $A > 0$ as given. The technology menu $G(B, A)$ is obtained as an envelope, by plugging the optimal choices $(K^*(B, A), L^*(B, A))$ into the local function. Formally, we write:

$$G(B, A) = \max_{(K,L)\in\Omega_\Phi} F(BK, AL) \qquad s.t. \qquad \Omega_\Phi = \{(K, L) \in \mathbb{R}^2_+ : \Phi(K, L) = 1\}. \tag{2}$$

In the basic treatment of the static problem (2), it is assumed that – alike the local function F – the aggregate function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous. Due to homogeneity, we may rewrite Φ in its intensive form, $\Phi(K, L) = \Phi\left(\frac{K}{L}, 1\right) L = \phi(k)L$. The aggregate (long-run, endogenous-technology) function Φ is interpreted as the aggregate production function faced by a firm or the aggregate utility function of a consumer. The difference between the local function F and the aggregate function Φ is that the former maps the quantities of inputs into an output keeping factor-specific technologies fixed, whereas the latter allows them to be chosen optimally. Under the production function interpretation, it is therefore natural to think of the local function as a short-run production function, and of the aggregate function – as a long-run one (León-Ledesma and Satchi, 2016). Again, economic interpretation of the aggregate function implies that in typical applications, it should be concave in each of its arguments.

In a slight generalization of the above problem, we replace the homogeneous function Φ with its homothetic counterpart, $\Phi_h = \phi_h \circ \Phi$, where $\phi_h : \mathbb{R}_+ \to \mathbb{R}$ is monotone (typically, increasing) and twice continuously differentiable.

2.3 Homotheticity, Additive Separability and Bergson's Theorem

Homotheticity of the considered functions has its profound consequences. Importantly, ever since Bergson (Burk, 1936) we know that every homothetic and additively separable function must be either of the Cobb–Douglas or of the CES form. In the symbols of our current study, Bergson's theorem can be stated as follows:

Theorem 1 (Bergson, 1936) *Let $F_h : \mathbb{R}^2_+ \to \mathbb{R}$ be a homothetic function which can be written* *as additively separable after a monotone transformation:*

$$\exists(f_h : \mathbb{R}_+ \to \mathbb{R},\ F : \mathbb{R}_+^2 \to \mathbb{R}_+) \qquad F_h(x, y) = f_h(F(x, y)), \tag{3}$$

$$\exists(f_s : \mathbb{R}_+ \to \mathbb{R},\ D_x, D_y : \mathbb{R}_+ \to \mathbb{R}) \qquad F_h(x, y) = f_s(D_x(x) + D_y(y)), \tag{4}$$

where f_h, f_s, D_x, D_y are monotone differentiable functions and F is an increasing, twice differentiable homogeneous function. Then either

$$D_x(x) = \alpha \ln x + c_x,\ D_y(y) = \beta \ln y + c_y \ \Rightarrow\ F(x, y) = c \cdot x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}}, \tag{5}$$

where α, β, c_x, c_y are arbitrary constants and $c = \exp\left(\frac{c_x+c_y}{\alpha+\beta}\right)$, or

$$D_x(x) = \alpha x^\rho + c_x,\ D_y(y) = \beta y^\rho + c_y \ \Rightarrow\ F(x, y) = (\alpha x^\rho + \beta y^\rho)^{\frac{1}{\rho}}, \tag{6}$$

where α, β, c_x, c_y are arbitrary constants and $\rho \neq 0$.

Proof. See Burk (1936) or (Rader, 1972, Theorem 8, page 212). ■

Bergson's theorem is the fundamental analytical cause why studies aiming at generalizing the CES framework must either give up homotheticity (e.g., Zhelobodko, Kokovin, Parenti, and Thisse, 2012) or additive separability (e.g., Revankar, 1971; Growiec and Mućk, 2016, this paper). It follows that in all the non-CES cases covered by the current study, the functions F, G and Φ cannot be written down as additively separable after any monotone transformation, a property shared among others by isoelastic elasticity of substitution (IEES) functions defined in Growiec and Mućk (2016).

Bergson's theorem is also the decisive reason why technology choice problems where the technology menu is constructed from *independent* idea distributions are necessarily limited to the Cobb–Douglas and CES cases (Growiec, 2008b, Proposition 3). Extending this result, in the current paper we demonstrate that this finding generalizes to a much broader case where the idea distributions are dependent and the dependence is modeled with some Archimedean copula. A thorough discussion of this issue will be provided in section 6. Here, it suffices to say that all non-trivial generalizations of existing results which will be provided in this paper, for which the technology menu G is constructed as a level curve of a certain joint distribution, require that the dependence between the dimensions is *not* modeled by any Archimedean copula.

2.4 Discussions and Clarifications

Let us clarify a few important concepts before we present our main results.

Generalized duality. Problems (1) and (2) are dual to one another, although not in the standard, linear sense of duality. Instead they are dual when taking the local function F as a (typically nonlinear) linking function. This generalized form of duality ("F-duality") encompasses linear duality as a special case (after a switch from maximization to minimization in the dual problem). At the same time, it also generalizes *idempotent duality*, where the linking function is a minimum function (Rubinov and Glover, 1998; Matveenko and Matveenko, 2015).[8] The latter case can be viewed as

[8] The term "idempotent duality" belongs to the realm of *tropical mathematics*. I am grateful to Matveenko and Matveenko (2015) for acquainting me with this notion. I was however deeply disappointed when I learned that *tropical mathematics* has nothing to do with *polar coordinates*.

a limiting case of F-duality.

Partial elasticities. Partial elasticities of homogeneous functions F, G and Φ with respect to their first arguments are defined as:

$$\pi_F(bk) = \frac{\partial F}{\partial (BK)}(BK, AL)\frac{BK}{F(BK, AL)} = \frac{f'(bk)bk}{f(bk)} > 0, \tag{7}$$

$$\pi_G(b) = \frac{\partial G}{\partial B}(B, A)\frac{B}{G(B, A)} = \frac{g'(b)b}{g(b)} > 0, \tag{8}$$

$$\pi_\Phi(k) = \frac{\partial \Phi}{\partial K}(K, L)\frac{K}{\tilde{\Phi}(K, L)} = \frac{\phi'(k)k}{\phi(k)} > 0. \tag{9}$$

Homogeneity implies that $\pi \in [0, 1]$ for all three functions and that partial elasticities with respect to their second arguments are equal to $1 - \pi$. It is also useful to define the *relative elasticities* Π, strictly increasing in π, as

$$\Pi_F(bk) = \frac{\pi_F(bk)}{1 - \pi_F(bk)} > 0, \qquad \Pi_G(b) = \frac{\pi_G(b)}{1 - \pi_G(b)} > 0, \qquad \Pi_\Phi(bk) = \frac{\pi_\Phi(k)}{1 - \pi_\Phi(k)} > 0. \tag{10}$$

Curvature. We define the *curvature* of homogeneous functions F, G and Φ as:

$$\theta_F(bk) = -\frac{f''(bk)bk}{f'(bk)}, \qquad \theta_G(bk) = -\frac{g''(b)b}{g'(b)}, \qquad \theta_\Phi(k) = -\frac{\phi''(k)k}{\phi'(k)}. \tag{11}$$

Hence, our measure of curvature is the Arrow–Pratt coefficient of relative risk aversion, also called the relative love of variety (Zhelobodko, Kokovin, Parenti, and Thisse, 2012). The curvature $\theta(x)$ is inversely linked to the elasticity of substitution $\sigma(x)$ via

$$\theta(x) = \frac{1 - \pi(x)}{\sigma(x)}. \tag{12}$$

As compared to the elasticity of substitution, the curvature $\theta(x)$ is relatively better suited to the simultaneous analysis of concave as well as convex functions: the curvature is always positive ($\theta(x) > 0$ for all x) for concave functions, always negative ($\theta(x) < 0$ for all x) for convex functions, and the curvature of linear functions is zero. Hence, when we think of concave production or utility functions, we ought to consider the case where $\theta(x) > 0$ in the entire domain.

Normalization. We carry out our analysis in *normalized units*. Production function normalization has been shown to be crucial for obtaining clean identification of the role of each parameter of the CES function (de La Grandville, 1989; Klump and de La Grandville, 2000; Klump, McAdam, and Willman, 2012). Its usefulness has also been demonstrated beyond the CES class (Growiec and Mućk, 2016) as well as for factor-specific technology choice problems (Growiec, 2013).

To maintain normalization while economizing on notation, we assume that K, L, B, A, k and b are already given in normalized units[9]:

$$K = \frac{\tilde{K}}{\tilde{K}_0}, \qquad L = \frac{\tilde{L}}{\tilde{L}_0}, \qquad B = \frac{\tilde{B}}{\tilde{B}_0}, \qquad A = \frac{\tilde{A}}{\tilde{A}_0}, \qquad k = \frac{\tilde{k}}{\tilde{k}_0}, \qquad b = \frac{\tilde{b}}{\tilde{b}_0}. \tag{13}$$

[9] In empirical studies, variables are often normalized around sample means (Klump, McAdam, and Willman, 2007, 2012).

Output is normalized in the same way as the inputs. We posit that $\tilde{G}(\tilde{B}_0, \tilde{A}_0) = G_0 \iff G(1,1) = 1$ as well as $\tilde{\Phi}(\tilde{K}_0, \tilde{L}_0) = \Phi_0 \iff \Phi(1,1) = 1$. Thus the level curves are

$$\Omega_G = \{(B,A) \in \mathbb{R}^2_+ : G(B,A) = 1\} = \{(\tilde{B},\tilde{A}) \in \mathbb{R}^2_+ : \tilde{G}(\tilde{B},\tilde{A}) = G_0\}, \tag{14}$$

$$\Omega_\Phi = \{(K,L) \in \mathbb{R}^2_+ : \Phi(K,L) = 1\} = \{(\tilde{K},\tilde{L}) \in \mathbb{R}^2_+ : \tilde{\Phi}(\tilde{K},\tilde{L}) = \Phi_0\}. \tag{15}$$

We also normalize the partial elasticities of the considered functions F, G and Φ:

$$\pi_{0F} \equiv \frac{\partial \tilde{F}}{\partial(\tilde{B}\tilde{K})}(\tilde{B}_0\tilde{K}_0, \tilde{A}_0\tilde{L}_0)\frac{\tilde{B}_0\tilde{K}_0}{\tilde{F}(\tilde{B}_0\tilde{K}_0, \tilde{A}_0\tilde{L}_0)} = \frac{f'(1)\cdot 1}{f(1)} = f'(1), \tag{16}$$

$$\pi_{0G} \equiv \frac{\partial \tilde{G}}{\partial \tilde{B}}(\tilde{B}_0, \tilde{A}_0)\frac{\tilde{B}_0}{\tilde{G}(\tilde{B}_0, \tilde{A}_0)} = \frac{g'(1)\cdot 1}{g(1)} = g'(1), \tag{17}$$

$$\pi_{0\Phi} \equiv \frac{\partial \tilde{\Phi}}{\partial \tilde{K}}(\tilde{K}_0, \tilde{L}_0)\frac{\tilde{K}_0}{\tilde{\Phi}(\tilde{K}_0, \tilde{L}_0)} = \frac{\phi'(1)\cdot 1}{\phi(1)} = \phi'(1). \tag{18}$$

In the discussion of our examples, we will pay special attention to the case where $\pi_{0F} = \pi_{0G} = \pi_{0\Phi}$. Such coincidence cannot be guaranteed for arbitrary functions, but it leads to particularly transparent outcomes whenever it happens to hold.

Relation to the problem of output/utility maximization subject to a budget constraint. It can be noticed that the primal factor-specific technology choice problem (1) considered in the current study has a similar structure to the classic problem (Shephard, 1953; Diewert, 1974; Fuss and McFadden, 1980) of output/utility maximization subject to a budget constraint (which leads to the construction of an envelope cost function as in (19)), whereas our dual problem (2) resembles the classic dual problem of cost minimization subject to a budget constraint viewed as a function of the prices r and w, (20):

$$C(r,w) = \max_{(K,L)\in\Omega_{B1}} Y(K,L) \qquad s.t. \qquad \Omega_{B1} = \{(K,L) \in \mathbb{R}^2_+ : rK + wL = 1\}, \tag{19}$$

$$F(K,L) = \min_{(r,w)\in\Omega_{B2}} C(r,w) \qquad s.t. \qquad \Omega_{B2} = \{(r,w) \in \mathbb{R}^2_+ : rK + wL = 1\}. \tag{20}$$

There are however differences between both setups: (i) the function linking quantities and prices (the budget constraint) is assumed to be linear here (and not an arbitrary local function F as in our more general setup), (ii) in line with the different economic interpretation but without any impact on the outcomes, the objectives and the constraints have switched places, (iii) to maintain consistency with the economic interpretation as well as with the underlying assumptions that the production/utility function is increasing with its arguments and the envelope cost function is decreasing, maximization is replaced with minimization in the dual problem.

Although mathematically similar, both problems are "orthogonal" in the sense that the factor-specific technology choice problem abstracts from factor prices and, symmetrically, the standard output/utility maximization problem abstracts from factor quality. This orthogonality property turns out to play a crucial role when we merge both problems into a unique problem of simultaneous factor-specific technology choice and output/utility maximization:

$$C(r,w) = \max_{(K,L)\in\Omega_{B1},(B,A)\in\Omega_G} F(BK,AL) \quad s.t. \quad \Omega_{B1} = \{(K,L) \in \mathbb{R}^2_+ : rK + wL = 1\}, \quad \Omega_G = \{(B,A) \in \mathbb{R}^2_+ : G(B,A) = 1\}. \tag{21}$$

This is a problem where the decision maker is allowed to choose both her favorite technology (subject to the given technology menu) and factor quantities (subject to the given budget constraint) at the same time (cf. León-Ledesma and Satchi, 2016). Inserting these optimal choices for all possible configurations of factor prices permits to construct – instead of the aggregate function taking factor quantities K and L as given – the envelope cost function which depends, in turn, only on the prices r and w.

The associated dual problem can be written as:

$$G(B,A) = \min_{(K,L)\in\Omega_F,(r,w)\in\Omega_C} rK + wL \quad s.t. \quad \Omega_F = \{(K,L)\in\mathbb{R}^2_+ : F(BK,AL)=1\}, \quad \Omega_C = \{(r,w)\in\mathbb{R}^2_+ : C(r,w)=1\}. \tag{22}$$

First order conditions for the joint and the separated optimization problems exactly coincide, underscoring the aforementioned orthogonality property: factor-specific technology choice and output/utility maximization, even when solved simultaneously, are not interdependent. It follows that – as long as factor quality does not enter the budget constraint and factor prices do not enter the technology menu – it is instructive to study the factor-specific technology choice problem separately as we do below.[10] Allowing for interdependence is left for future research.

Relation to the literature on factor-augmenting technical change. The discussed setup is static and thus abstracts from technical change which – by definition – happens over time. Moreover, the technological underpinnings of the economy are in fact not only constant but also invisible because in the normalization procedure, the current overall Hicks-neutral technology level of the economy has been conveniently incorporated in F_0, G_0 and Φ_0, whereas the current relative productivity of both factors has been included in π_{OF}, π_{0G} and $\pi_{0\Phi}$.

However, the possibility of explicit technical change can be incorporated as an extension of our setup by conditioning at least two of the three functions F, G or Φ on time. In particular, if one wants to consider *factor-augmenting* technical change (which can be decomposed into Hicks-neutral technical change and the bias in technical change, working in favor of one of the factors),[11] one has to replace either:

- $F(BK,AL)$ with $F(\lambda_K BK,\lambda_L AL)=\lambda_L F(\lambda_k BK,AL)$, or
- $G(B,A)$ with $G(\lambda_K B,\lambda_L A)=\lambda_L G(\lambda_k B,A)$, or
- $\Phi(K,L)$ with $\Phi(\lambda_K K,\lambda_L L)=\lambda_L\Phi(\lambda_k K,L)$,

where the variation in $\lambda_K>0$ and $\lambda_L>0$ over time represents capital- and labor-augmenting technical change, respectively. Equivalently, changes in λ_L can be said to represent Hicks-neutral technical change, and then $\lambda_k=\frac{\lambda_K}{\lambda_L}$ measures the capital bias in technical change.[12] Adding a

[10]Additional second order conditions may be needed, however, to ensure the existence of an interior solution to the joint problem (León-Ledesma and Satchi, 2016, Appendix A.2).

[11]See, e.g., Acemoglu (2002, 2003); Klump, McAdam, and Willman (2007); León-Ledesma, McAdam, and Willman (2010).

[12]Growiec (2008a) studies factor-specific technology choice in a dynamic framework with Hicks-neutral technical change. Growiec (2013) allows for biased technical change and discusses the emerging possibility of a difference between the direction of R&D (which only affects the shape of the technology menu G) and the direction of technical change (which also incorporates firms' optimal technology choices). Biased technical change is also allowed within factor-specific technology choice frameworks studied by León-Ledesma and Satchi (2016), an estimated business–cycle model with a short-run CES and a long-run Cobb–Douglas technology which thus circumvents the

dynamic edge to the considered framework remains an important task which we leave for further research.

3 Optimal Technology Choice

3.1 The Primal Problem

To solve the primal optimization problem for a given pair (K, L), we set up the following Lagrangian $\mathcal{L}_P$:

$$\mathcal{L}_P(B, A) = F(BK, AL) + \lambda(G(B, A) - 1). \tag{23}$$

We find that as long as the curvature of the local function F exceeds the curvature of the technology menu G (i.e., there are relatively few substitution possibilities along the local function), there exists a unique interior solution to the problem which equalizes partial elasticities of the local function and the technology menu. We also find that the optimal technology choice is biased towards the abundant factor ($\frac{\partial b^*(k)}{\partial k} > 0$) if factors are gross substitutes along a concave local function or if the local function is convex ($1 - \pi_F(bk) - \theta_F(bk) > 0$, which requires that $\sigma_F(bk) > 1$ or $\sigma_F(bk) < 0$). Otherwise, optimal technology choice is biased towards the scarce factor ($\frac{\partial b^*(k)}{\partial k} < 0$). Then factors are gross complements along a concave local function ($\sigma_F(bk) \in (0, 1)$). In the intermediate, knife-edge case where the local technology is Cobb–Douglas ($\sigma_F(bk) = 1$), optimal technology choice does not depend on factor endowments, i.e., $b^*(k)$ is constant.

Theorem 2 *Let $F, G : \mathbb{R}^2_+ \to \mathbb{R}_+$ be increasing, twice continuously differentiable homogeneous functions satisfying $\theta_F(b^*(k)k) > \theta_G(b^*(k))$ for a given pair $(K, L) \in \mathbb{R}^2_+$, and excluding the case where both of them are Cobb–Douglas functions. Then the problem (1) allows a unique interior maximum where*

$$\Pi_F(b^*(k)k) = \Pi_G(b^*(k)), \tag{24}$$

and

$$B^*(k) = \frac{b^*(k)}{g(b^*(k))}, \qquad A^*(k) = \frac{1}{g(b^*(k))}. \tag{25}$$

The partial elasticity of the optimal technology choice $b^*(k)$ equals:

$$\frac{\partial b^*(k)}{\partial k} \frac{k}{b^*(k)} = \frac{1 - \pi_F(bk) - \theta_F(bk)}{\theta_F(bk) - \theta_G(b)}. \tag{26}$$

Proof. Equation (24) is obtained directly from the two first order conditions for the Lagrangian by eliminating λ. Equation (25) follows from the fact that along the technology menu, $G(B, A) = g(b)A = 1$.

To ascertain that the found solution is indeed a maximum, we compute the second order

Steady State Growth Theorem (Uzawa, 1961) and reconciles the long-run balanced growth requirement with gross complementarity of both factors and non-neutral technical change; and Growiec, McAdam, and Mućk (2015), a calibrated model of medium-to-long run swings.

conditions, which imply that:

$$\frac{\partial^2 \mathcal{L}_P}{\partial B^2} = \frac{K}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right), \tag{27}$$

$$\frac{\partial^2 \mathcal{L}_P}{\partial A^2} = \frac{b^2 K}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right), \tag{28}$$

$$\frac{\partial^2 \mathcal{L}_P}{\partial B \partial A} = -\frac{bK}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right), \tag{29}$$

and thus $\frac{\partial^2 \mathcal{L}_P}{\partial K^2} < 0$ and $\frac{\partial^2 \mathcal{L}_P}{\partial L^2} < 0$ if and only if $\theta_F(b^*(k)k) > \theta_G(b^*(k))$. Though the Hessian is equal to zero because F and G are homogeneous functions (Moysan and Senouci, 2016), concavity is guaranteed along the tangent to the constraint, i.e., along the line

$$\left\{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}^2 : [g'(b) \quad g(b) - bg'(b)] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = 0\right\}. \tag{30}$$

Indeed, for all $h_1 \neq 0$ we obtain:

$$\begin{bmatrix} h_1 & -\frac{\Pi_G}{b}h_1 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \mathcal{L}_P}{\partial B^2} & \frac{\partial^2 \mathcal{L}_P}{\partial B \partial A} \\ \frac{\partial^2 \mathcal{L}_P}{\partial B \partial A} & \frac{\partial^2 \mathcal{L}_P}{\partial A^2} \end{bmatrix} \begin{bmatrix} h_1 \\ -\frac{\Pi_G}{b}h_1 \end{bmatrix} = h_1^2 \frac{K}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right)(1 + \Pi_G^2) < 0. \tag{31}$$

Let us also rewrite (24) as:

$$X_P(b,k) = \Pi_F(bk) - \Pi_G(b) = 0. \tag{32}$$

Using the implicit function theorem and the equality $\pi = \pi_F(bk) = \pi_G(b)$ (which follows from (24)), we obtain:

$$\begin{aligned} \frac{\partial b^*(k)}{\partial k} &= -\frac{\frac{\partial X_P}{\partial k}}{\frac{\partial X_P}{\partial b}} = \frac{\frac{\partial \Pi_F}{\partial (bk)}b}{\frac{\partial \Pi_G}{\partial b} - \frac{\partial \Pi_F}{\partial (bk)}k} = \frac{\frac{f'(bk)b}{f(bk)} \frac{1-\pi_F(bk)-\theta_F(bk)}{(1-\pi_F(bk))^2}}{\frac{g'(b)}{g(b)} \frac{1-\pi_G(b)-\theta_G(b)}{(1-\pi_G(bk))^2} - \frac{f'(bk)k}{f(bk)} \frac{1-\pi_F(bk)-\theta_F(bk)}{(1-\pi_F(bk))^2}} = \\ &= \frac{b}{k}\left(\frac{1-\pi-\theta_F(bk)}{\theta_F(bk)-\theta_G(b)}\right), \end{aligned} \tag{33}$$

or (26). Uniqueness of the optimum $b^*(k)$ follows from the fact that (unless $\frac{\partial \Pi_G}{\partial b} = \frac{\partial \Pi_F}{\partial (bk)} = 0$ which happens only in the excluded case where F and G are Cobb–Douglas functions) the denominator in (33) is positive. ■

Theorem 2 can be straightforwardly generalized to the case of homothetic functions. Intuitively, this is due to the fact that level curves of any function have exactly the same shape whether or not it has been subjected to a monotone transformation.

Theorem 3 *Let $F_h, G_h : \mathbb{R}_+^2 \to \mathbb{R}_+$ be increasing, twice continuously differentiable homothetic functions such that $F_h = f_h \circ F$ and $G_h = g_h \circ G$ where $f_h, g_h : \mathbb{R}_+ \to \mathbb{R}$ are increasing, twice continuously differentiable functions, and F and G are as in Theorem 2. Then the problem*

$$\Phi_h(K,L) = \max_{(B,A)\in\Omega_G} F_h(BK, AL) \qquad s.t. \qquad \Omega_G = \{(B,A) \in \mathbb{R}_+^2 : G_h(B,A) = g_h(1)\}. \tag{34}$$

allows a unique interior maximum satisfying (24), (25) and (26).

Proof. First, we observe that $G_h(B,A) = g_h(G(B,A)) = g_h(1) \iff G(B,A) = 1$. Thus the technology menu is exactly the same as in (1). We then repeat all the steps of proof of Theorem 2 and observe that all terms related to $f_h'(\cdot)$ and $f_h''(\cdot)$ cancel out in the first and second order conditions, respectively. ■

An analogous result can be formulated for the case where one or both of the functions f_h, g_h is decreasing. The only caveat is that when f_h is decreasing, maximization in (1) should be replaced with minimization.

Finally, we observe that the obtained outcome depends on the underlying homogeneous functions F and G only and not on their monotone transformations f_h and g_h. Therefore, not only is it more convenient to present the main outcomes in terms of homogeneous functions, but it also comes without loss of generality.

3.2 The Dual Problem

The construction of the dual problem is extremely similar to its primal counterpart, so the results are very much alike as well. To solve the dual optimization problem for a given pair (B, A), we set up the following Lagrangian $\mathcal{L}_D$:

$$\mathcal{L}_D(K,L) = F(BK, AL) + \lambda(\Phi(K,L) - 1). \tag{35}$$

Fully analogously to the primal case, we find that as long as the curvature of the local function F exceeds the curvature of the aggregate function Φ (i.e., there are less substitution possibilities locally than globally), there exists a unique interior solution to the problem which equalizes partial elasticities of the local and the aggregate function. We also find that the choice of the factor ratio is biased towards the more productive factor (or factor with superior quality, $\frac{\partial k^*(b)}{\partial b} > 0$) if factors are gross substitutes along a concave local function or if the local function is convex. Conversely, if factors are gross complements along a concave local function, factor choice is biased towards the less productive factor. If the local technology is Cobb–Douglas then the optimal factor choice does not depend on the technology endowment, i.e., $k^*(b)$ is constant.

Proofs of the following theorems are fully symmetric to the proofs of Theorems 2 and 3 and therefore have been omitted.

Theorem 4 *Let $F, \Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ be increasing, twice continuously differentiable homogeneous functions satisfying $\theta_F(bk^*(b)) > \theta_\Phi(k^*(b))$ for a given pair $(B,A) \in \mathbb{R}^2_+$, and excluding the case where both of them are Cobb–Douglas functions. Then the problem (2) allows a unique interior maximum where*

$$\Pi_F(bk^*(b)) = \Pi_\Phi(k^*(b)), \tag{36}$$

and

$$K^*(b) = \frac{k^*(b)}{\phi(k^*(b))}, \qquad L^*(b) = \frac{1}{\phi(k^*(b))}. \tag{37}$$

The partial elasticity of the optimal factor choice $k^*(b)$ equals:

$$\frac{\partial k^*(b)}{\partial b}\frac{b}{k^*(b)} = \frac{1 - \pi_F(bk) - \theta_F(bk)}{\theta_F(bk) - \theta_\Phi(k)}. \tag{38}$$

Theorem 5 *Let $F_h, \Phi_h : \mathbb{R}^2_+ \to \mathbb{R}_+$ be increasing, twice continuously differentiable homothetic functions such that $F_h = f_h \circ F$ and $\Phi_h = \phi_h \circ \Phi$ where $f_h, \phi_h : \mathbb{R}_+ \to \mathbb{R}$ are increasing, twice continuously differentiable functions, and F and Φ are as in Theorem 4. Then the problem*

$$G_h(B, A) = \max_{(K,L)\in\Omega_\Phi} F_h(BK, AL) \qquad s.t. \qquad \Omega_\Phi = \{(K, L) \in \mathbb{R}^2_+ : \Phi_h(K, L) = \phi_h(1)\}. \tag{39}$$

allows a unique interior maximum satisfying (36), (37) and (38).

Once again, an analogous result to Theorem 5 can also be formulated for the case where one or both of the functions f_h, ϕ_h is decreasing; when f_h is decreasing, maximization in (2) should be replaced with minimization.

Having identified the optimal choices in the primal and dual problem, we are now in a position to insert them into the local function and thus to construct the appropriate envelopes.

4 The Aggregate Function and the Technology Menu: Construction and Duality

As stated above, the aggregate function Φ is constructed as an envelope of local functions by inserting the optimal technology choices from the primal problem, as derived in Theorem 2, into the local function F. Symmetrically, the technology menu G is constructed as an envelope of local functions by inserting the optimal factor choices from the dual problem, as derived in Theorem 4. The domains of both envelopes include all arguments for which interior optimal choices exist, i.e., all arguments for which the curvature of the local function exceeds the one of the constraint. The resultant envelopes have the following properties.

Theorem 6 *Let $\mathcal{D}_\Phi = \{(K, L) \in \mathbb{R}^2_+ : \theta_F(b^*(k)k) > \theta_G(b^*(k))\}$ and $\mathcal{D}_G = \{(B, A) \in \mathbb{R}^2_+ : \theta_F(bk^*(b)) > \theta_\Phi(k^*(b))\}$ where $b^*(k)$ solves (24) and $k^*(b)$ solves (36). Then there exists a unique increasing homogeneous* aggregate function *$\Phi : \mathcal{D}_\Phi \to \mathbb{R}_+$ solving problem (1) as well as a unique increasing homogeneous* technology menu *$G : \mathcal{D}_G \to \mathbb{R}_+$ solving problem (2). Their respective intensive forms are given by:*

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))}, \qquad g(b) = \frac{f(bk^*(b))}{\phi(k^*(b))}. \tag{40}$$

Proof. Existence and uniqueness of Φ solving problem (1) follows from Theorem 2. It also follows that

$$\phi(k) = \frac{\Phi(K, L)}{L} = \frac{F(B^*(k)K, A^*(k)L)}{L} = f(b^*(k)k)A^*(k) = \frac{f(b^*(k)k)}{g(b^*(k))}. \tag{41}$$

Existence and uniqueness of G solving problem (2) follows from Theorem 4. It also follows that

$$g(b) = \frac{G(B, A)}{A} = \frac{F(BK^*(B, A), AL^*(B, A))}{A} = f(bk^*(b))L^*(b) = \frac{f(bk^*(b))}{\phi(k^*(b))}. \tag{42}$$

Both functions are homogeneous by construction. Computing $\phi'(k)$ and $g'(b)$ from (40), using

(24), (26), (36), (38) and rearranging we obtain:

$$\frac{\phi'(k)}{\phi(k)} = \frac{f'(bk)}{f(bk)}\left(\frac{\partial b^*(k)}{\partial k}k + b^*(k)\right) - \frac{g'(b)}{g(b)}\frac{\partial b^*(k)}{\partial k} = \frac{\pi}{k} > 0, \tag{43}$$
$$\frac{g'(b)}{g(b)} = \frac{f'(bk)}{f(bk)}\left(b\frac{\partial k^*(b)}{\partial b} + k^*(b)\right) - \frac{\phi'(k)}{\phi(k)}\frac{\partial k^*(b)}{\partial b} = \frac{\pi}{b} > 0, \tag{44}$$

where in each case the positivity of π follows from assumption that the other two functions are increasing. Thus Φ and G are increasing. ■

Please note that while both functions g and ϕ are increasing in their arguments, the optimal choices $b^*(k)$ and $k^*(b)$ do not have to be monotone (and hence, bijective). Therefore the mutual duality ("F-duality") of the aggregate function and the technology menu must be limited to the domain where the optimal choices can be inverted.

It is no surprise that the aggregate function and the technology menu are dual to one another, and thus both equalities in (40) hold simultaneously, only on intervals where $b^*(k)$ and $k^*(b)$ are monotone, and thus can be inverted, so that $k = k^*(b^*(k))$ and $b = b^*(k^*(b))$, as well as:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))} = \frac{f(b^*(k)k^*(b^*(k)))}{g(b^*(k))} = \frac{f(b^*(k)k^*(b^*(k)))}{\frac{f(b^*(k)k^*(b^*(k)))}{\phi(k^*(b^*(k)))}} = \phi(k^*(b^*(k))), \tag{45}$$
$$g(b) = \frac{f(bk^*(b))}{\phi(k^*(b))} = \frac{f(b^*(k^*(b))k^*(b))}{\phi(k^*(b))} = \frac{f(b^*(k^*(b))k^*(b))}{\frac{f(b^*(k^*(b))k^*(b))}{g(b^*(k^*(b)))}} = g(b^*(k^*(b))). \tag{46}$$

Interestingly, however, these intervals coincide precisely with the domain in which both inputs are either (i) gross complements along a concave local function (with $\sigma_F(bk) \in (0,1)$ and $1 - \pi_F(bk) - \theta_F(bk) < 0$), or (ii) gross substitutes along a concave local function with the additional possibility of a convex local function (i.e., the case where $1 - \pi_F(bk) - \theta_F(bk) > 0$, requiring that either $\sigma_F(bk) > 1$ or $\sigma_F(bk) < 0$). Most of the production function studies thus far concentrated on the former possibility (e.g., Rubinov and Glover, 1998; Jones, 2005; Growiec, 2013; Matveenko and Matveenko, 2015) and assumed that factors are always gross complements along the local production function. We generalize these studies by accommodating both variants.

Theorem 7 *Let $\Omega = \{(k,b) \in \mathcal{D}_\Phi \times \mathcal{D}_G : 1 - \pi_F(bk) - \theta_F(bk) \neq 0\}$. Then for each connected subset of Ω, both equalities in (40) hold simultaneously and partial elasticities of F, G and Φ are equal:*

$$\pi = \pi_F(bk) = \pi_G(b) = \pi_\Phi(k). \tag{47}$$

For all $(k,b) \in \Omega$ it also holds that $1 - \pi - \theta_G(b) = 0 \iff 1 - \pi - \theta_\Phi(k) = 0$ and otherwise the curvatures of the three functions are linked via

$$\frac{1}{1-\pi-\theta_F(bk)} = \frac{1}{1-\pi-\theta_G(b)} + \frac{1}{1-\pi-\theta_\Phi(k)}. \tag{48}$$

Proof. Equation (47) follows from (24) and (36) in the case where both of them hold at the same time. Moreover, when $1 - \pi_F(bk) - \theta_F(bk) \neq 0$, we can insert (38) into (26), use (47) and obtain:

$$\frac{1-\pi-\theta_F(bk)}{\theta_F(bk)-\theta_G(b)} = \frac{\theta_F(bk)-\theta_\Phi(k)}{1-\pi-\theta_F(bk)}, \tag{49}$$

and hence,

$$(1-\pi-\theta_G(b))(1-\pi-\theta_\Phi(k)) = (1-\pi-\theta_F(bk))(1-\pi-\theta_G(b))+(1-\pi-\theta_F(bk))(1-\pi-\theta_\Phi(k)). \quad (50)$$

It follows that $1-\pi-\theta_G(b)=0 \iff 1-\pi-\theta_\Phi(k)=0$ and if both terms are nonzero, then we can divide both sides of equation (50) by $(1-\pi_F(bk)-\theta_F(bk))(1-\pi-\theta_G(b))(1-\pi-\theta_\Phi(k))$, yielding (48). ■

Equation (48) is a precise quantitative description of the relationship between the curvatures of the local function, the technology menu and the aggregate function. It also has some very intuitive properties. First, we have that $\theta_F(bk)$ always exceeds both $\theta_G(b)$ and $\theta_\Phi(k)$. Hence, factor-specific technology choice always adds more flexibility to the local function, thereby decreasing its curvature (and thus, under concavity, increasing its elasticity of substitution (Growiec, 2013)).[13]

Second, it is instructive to evaluate the signs of both sides of (48). If the left-hand side is negative, meaning that factors are gross complements along the local function (by all means the usual case in the production function literature), then $1-\pi-\theta_G(b)$ and $1-\pi-\theta_\Phi(k)$ must be of opposing signs. Hence, it must be that either the technologies are gross substitutes along the technology menu but the factors are gross complements along the aggregate function, or vice versa, the technologies are gross complements along the technology menu and the factors are gross substitutes along the aggregate function. The remaining possibility is that the left-hand side of (48) is positive, so that the factors are gross substitutes already along the local function. In such a situation, both $1-\pi-\theta_G(b)$ and $1-\pi-\theta_\Phi(k)$ must be positive as well, encompassing the cases of gross substitutability and convexity.

So far we have demonstrated the construction and duality of the aggregate function and the technology menu for the homogeneous case. From Theorems 3 and 5 we know, however, that extrema of both considered problems are invariant under monotone transformations. This means that we can extend the scope of our study to homothetic functions.

More precisely, for every homothetic function $F_h = f_h \circ F$, where $f_h : \mathbb{R}_+ \to \mathbb{R}$ is monotone and twice continuously differentiable and $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous, from (40) we have that

$$\tilde{\Phi}_h(K,L) = F_h(B^*(k)K, A^*(k)L) = f_h(f(b^*(k)k)A^*(k)L) = f_h\left(\frac{f(b^*(k)k)}{g(b^*(k))}L\right) = f_h(\Phi(K,L)), \quad (51)$$

$$\tilde{G}_h(B,A) = F_h(BK^*(b), AL^*(b)) = f_h(f(bk^*(b))AL^*(b)) = f_h\left(\frac{f(bk^*(b))}{\phi(k^*(b))}A\right) = f_h(G(B,A)). \quad (52)$$

This leads to the construction of $\tilde{\Phi}_h = f_h \circ \Phi$ from problem (1) and of $\tilde{G}_h = f_h \circ G$ from problem (2). Clearly, both functions are homothetic. They are also dual to one another in the sense that maximizing $F_h(BK, AL)$ subject to $g_h(G(B,A)) = g_h(1)$ leads to the construction of $\tilde{\Phi}_h(K,L)$ for any monotone function $g_h : \mathbb{R}_+ \to \mathbb{R}$, and maximizing $F_h(BK, AL)$ subject to $\phi_h(\Phi(K,L)) = \phi_h(1)$ leads to the construction of $\tilde{G}_h(B,A)$ for any monotone function $\phi_h : \mathbb{R}_+ \to \mathbb{R}$. However, inclusion of g_h and ϕ_h in the above formulas underscores that allowing for monotone transformations of the homogeneous functions F, G, Φ compromises uniqueness of the resulting functions. Indeed, the results are unchanged also when we replace $\tilde{\Phi}_h$ with Φ_h (i.e., f_h with an arbitrary ϕ_h) or $\tilde{G}_h$ with

[13] A similar relationship has also been derived by León-Ledesma and Satchi (2016) in their equation (15). Their equation is however somewhat less transparent because they use an implicit specification of the technology menu and solve the primal problem only.

G_h (i.e., f_h with an arbitrary g_h).

In sum, we have shown that when the local function F, the technology menu G and the aggregate function Φ are homogeneous (CRS) functions, then the respective solutions to (1) and (2) are unique and – as long as the optimal choice is also invertible – mutually dual. However, if these functions are not homogeneous but only homothetic then the respective solutions still exist and are still mutually dual (subject to the same invertibility requirement), but they are specified only up to a monotone transformation and thus are no longer unique.

5 Notable Special Cases

Several special parametrizations of the above general setup have already been discussed in the literature. We shall now provide an overview of these cases, thus underscoring the wide applicability of our general theorems. Most notably, under certain assumptions they can be used as microfoundation for aggregate Cobb–Douglas and CES production/utility functions. Like in the earlier sections, we will first explain the results in the homogeneous case and only then generalize them to the homothetic case.

5.1 The Cobb–Douglas Function

The primal problem with a Cobb–Douglas technology menu has been studied by, among others, Jones (2005) and León-Ledesma and Satchi (2016). Its variant with a Cobb–Douglas local function has been reviewed as an example in Growiec (2008a). The appendix to Growiec (2013) has also considered the case of a continuum of factors. Here we reproduce these results as special cases of our general theory as well as elucidate certain important problems which may arise in the primal and dual problems under this particular parametrization.

Cobb–Douglas local function. If the local function is of the homogeneous, normalized Cobb–Douglas form, then:

$$F(BK, AL) = (BK)^{\pi_{0F}}(AL)^{1-\pi_{0F}}, \qquad f(bk) = (bk)^{\pi_{0F}}. \tag{53}$$

Assuming that G is not Cobb–Douglas and that $\theta_G(b) < 1 - \pi_{0F}$, from (24) we obtain that the optimal technology choice is independent of k:

$$\Pi_{0F} = \Pi_G(b) \Rightarrow b^*(k) \equiv b^* = \Pi_G^{-1}(\Pi_{0F}). \tag{54}$$

Inserting this choice for all $(K, L) \in \mathbb{R}^2_+$, from (40) we obtain:

$$\phi(k) = \frac{f(b^*k)}{g(b^*)} = \left(\frac{(b^*)^{\pi_{0F}}}{g(b^*)}\right) k^{\pi_{0F}} \Rightarrow \Phi(K, L) = \left(\frac{(b^*)^{\pi_{0F}}}{g(b^*)}\right) K^{\pi_{0F}} L^{1-\pi_{0F}}. \tag{55}$$

It means that irrespective of the shape of G, the aggregate function must be Cobb–Douglas with the same exponent π_{0F} as the local function.

If, additionally, $\pi_{0F} = \pi_{0G}$ then $b^* = 1$ and hence the constant becomes equal to unity, implying $\Phi(K, L) = K^{\pi_{0F}} L^{1-\pi_{0F}}$.

A fully symmetric result is obtained when solving the dual problem with a Cobb–Douglas local function. In that case, assuming that Φ is not Cobb–Douglas and that $\theta_\Phi(k) < 1 - \pi_{0F}$, from (36) we obtain that the optimal factor choice is independent of b:

$$\Pi_{0F} = \Pi_\Phi(k) \Rightarrow k^*(b) \equiv k^* = \Pi_\Phi^{-1}(\Pi_{0F}). \tag{56}$$

Inserting this choice for all $(B, A) \in \mathbb{R}^2_+$, from (40) we obtain:

$$g(b) = \frac{f(bk^*)}{\phi(k^*)} = \left(\frac{(k^*)^{\pi_{0F}}}{\phi(k^*)}\right) b^{\pi_{0F}} \Rightarrow G(B, A) = \left(\frac{(k^*)^{\pi_{0F}}}{\phi(k^*)}\right) B^{\pi_{0F}} A^{1-\pi_{0F}}. \tag{57}$$

Again, if additionally $\pi_{0F} = \pi_{0\Phi}$ then $k^* = 1$ and hence $G(B, A) = B^{\pi_{0F}} A^{1-\pi_{0F}}$.

While intuitive, the case of Cobb–Douglas local functions is pathological in the sense that, because the optimal choice is constant and thus not invertible, the technology menu and the aggregate function cannot be viewed as dual objects. Indeed, trying to solve the primal problem when F and G are both Cobb–Douglas functions with the same exponent π_{0F}, immediately leads to indeterminacy:

$$\max_{(B,A)\in\mathbb{R}^2_+} F(BK, AL) = (BK)^{\pi_{0F}} (AL)^{1-\pi_{0F}} \qquad s.t. \qquad G(B, A) = B^{\pi_{0F}} A^{1-\pi_{0F}} = 1 \tag{58}$$

implies maximizing $K^{\pi_{0F}} L^{1-\pi_{0F}}$ which does not depend on B and A. Indeterminacy would also follow if we tried to solve the dual problem when F and Φ are both Cobb–Douglas with the same exponent π_{0F}.

This pathological outcome is a direct consequence of violation of the curvature assumption in Theorem 2 (when solving the primal problem while assuming that F and G are Cobb–Douglas functions with the same exponent) or in Theorem 4 (when making this assumption for F and Φ in the dual problem).

Cobb–Douglas technology menu. Let us now consider the case where the technology menu G is Cobb–Douglas with an exponent π_{0G}:

$$G(B, A) = B^{\pi_{0G}} A^{1-\pi_{0G}}, \qquad g(b) = b^{\pi_{0G}} \tag{59}$$

and the local function exhibits more curvature, $\theta_F(bk) > 1 - \pi_{0G}$. In this case, the optimal technology choice is monotone and thus duality is present again. From (24) we obtain

$$\Pi_F(b^*(k)k) = \Pi_{0G} \Rightarrow b^*(k) = \frac{\Pi_F^{-1}(\Pi_{0G})}{k}. \tag{60}$$

Inserting this choice for all $(K, L) \in \mathbb{R}^2_+$, from (40) we obtain:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))} = \left(\frac{f(\Pi_F^{-1}(\Pi_{0G}))}{(\Pi_F^{-1}(\Pi_{0G}))^{\pi_{0G}}}\right) k^{\pi_{0G}} \Rightarrow \Phi(K, L) = \left(\frac{f(\Pi_F^{-1}(\Pi_{0G}))}{(\Pi_F^{-1}(\Pi_{0G}))^{\pi_{0G}}}\right) K^{\pi_{0G}} L^{1-\pi_{0G}}. \tag{61}$$

It means that irrespective of the shape of F, the aggregate function must be Cobb–Douglas with the same exponent π_{0G} as the technology menu. If, additionally, $\pi_{0F} = \pi_{0G}$ then $b^*(k) = 1/k$ and hence the constant becomes equal to unity, implying $\Phi(K, L) = K^{\pi_{0G}} L^{1-\pi_{0G}}$.

Cobb–Douglas aggregate function. The dual problem for a Cobb–Douglas aggregate function $\Phi(K,L) = K^{\pi_{0\Phi}} L^{1-\pi_{0\Phi}}$ is solved analogously. From (36) we obtain

$$\Pi_F(bk^*(b)) = \Pi_{0\Phi} \Rightarrow k^*(b) = \frac{\Pi_F^{-1}(\Pi_{0\Phi})}{b}. \tag{62}$$

Inserting this choice for all $(K,L) \in \mathbb{R}_+^2$, from (40) we obtain:

$$g(b) = \frac{f(bk^*(b))}{\phi(k^*(b))} = \left(\frac{f(\Pi_F^{-1}(\Pi_{0\Phi}))}{(\Pi_F^{-1}(\Pi_{0\Phi}))^{\pi_{0\Phi}}} \right) b^{\pi_{0\Phi}} \Rightarrow G(B,A) = \left(\frac{f(\Pi_F^{-1}(\Pi_{0\Phi}))}{(\Pi_F^{-1}(\Pi_{0\Phi}))^{\pi_{0\Phi}}} \right) B^{\pi_{0\Phi}} A^{1-\pi_{0\Phi}}. \tag{63}$$

It means that irrespective of the shape of F, the technology menu must be Cobb–Douglas with the same exponent $\pi_{0\Phi}$ as the aggregate function. If, additionally, $\pi_{0F} = \pi_{0\Phi}$ then $k^*(b) = 1/b$ and hence the constant becomes equal to unity, implying $G(B,A) = B^{\pi_{0G}} A^{1-\pi_{0G}}$.

The homothetic case. Extending the above results, we shall now consider a homothetic local function:

$$F_h(BK, AL) = \ln\left((BK)^{\pi_{0F}} (AL)^{1-\pi_{0F}}\right) = \pi_{0F} \ln(BK) + (1-\pi_{0F}) \ln(AL). \tag{64}$$

This monotone transformation of a Cobb–Douglas local function is particularly often used in the modeling of consumer choices, where it represents additively separable logarithmic preferences. As argued in the previous sections, the results here are the same as for an untransformed Cobb–Douglas local function. Hence, while analytically convenient, such a specification is also very restrictive and in fact represents a pathological case where the optimal technology choice is independent of factor endowments and thus the technology menu and the aggregate function are not mutually dual.

5.2 The CES Function

The primal problem with a CES local function and a CES technology menu has been analyzed by, among others, Growiec (2008b, 2013). The former study also touched upon the dual problem, whereas the appendix to the latter considered the more general case of a continuum of factors.

It turns out that with a CES (or Leontief) local function, a CES technology menu is dual to a CES aggregate function – and vice versa. Let us now briefly review this case as a specific application of our general theory.

Formally, for the primal problem let us assume that

$$F(BK, AL) = \left(\pi_{0F}(BK)^\rho + (1-\pi_{0F})(AL)^\rho\right)^{\frac{1}{\rho}}, \qquad G(B,A) = \left(\pi_{0G} B^\alpha + (1-\pi_{0G}) A^\alpha\right)^{\frac{1}{\alpha}}, \tag{65}$$

with $\rho \neq 0$ and $\alpha \neq 0$ as well as $\rho < \alpha$ which implies $\theta_F(bk) > \theta_G(b)$. From (24) we obtain:

$$\Pi_{0F}(bk)^\rho = \Pi_{0G} b^\alpha \Rightarrow b^*(k) = \left(\frac{\Pi_{0F}}{\Pi_{0G}}\right)^{\frac{1}{\alpha-\rho}} k^{\frac{\rho}{\alpha-\rho}}. \tag{66}$$

Inserting this choice for all $(K,L) \in \mathbb{R}^2_+$, from (40) we obtain:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))} = \frac{\left(\pi_{0F}(\frac{\Pi_{0F}}{\Pi_{0G}})^{\frac{\rho}{\alpha-\rho}} k^{\frac{\alpha\rho}{\alpha-\rho}} + (1-\pi_{0F})\right)^{\frac{1}{\rho}}}{\left(\pi_{0G}(\frac{\Pi_{0F}}{\Pi_{0G}})^{\frac{\alpha}{\alpha-\rho}} k^{\frac{\alpha\rho}{\alpha-\rho}} + (1-\pi_{0G})\right)^{\frac{1}{\alpha}}} = \zeta \cdot \left(\pi_{0\Phi}k^{\xi} + (1-\pi_{0\Phi})\right)^{\frac{1}{\xi}}, \tag{67}$$

where $\xi = \frac{\alpha\rho}{\alpha-\rho}$ denotes the elasticity parameter of the resultant aggregate function (linked to its elasticity of substitution via $\sigma_\Phi = \frac{1}{1-\xi}$), the multiplicative constant equals $\zeta = (1-\pi_{0F})^{\frac{1}{\rho}}(1-\pi_{0G})^{-\frac{1}{\alpha}}(1-\pi_{0\Phi})^{-\frac{1}{\xi}}$, and $\pi_{0\Phi}$ is the partial elasticity of the aggregate function at the point of normalization which satisfies:

$$\Pi_{0F}^{\frac{1}{\rho}} = \Pi_{0G}^{\frac{1}{\alpha}}\Pi_{0\Phi}^{\frac{1}{\xi}} \qquad \Longleftrightarrow \qquad \left(\frac{\pi_{0F}}{1-\pi_{0F}}\right)^{\frac{1}{\rho}} = \left(\frac{\pi_{0G}}{1-\pi_{0G}}\right)^{\frac{1}{\alpha}}\left(\frac{\pi_{0\Phi}}{1-\pi_{0\Phi}}\right)^{\frac{1}{\xi}}. \tag{68}$$

We also observe that in the special case where $\pi_{0F} = \pi_{0G} = \pi_{0\Phi}$, the optimal technology choice simplifies to $b^*(k) = k^{\frac{\rho}{\alpha-\rho}}$ with $\zeta = 1$.

Hence, indeed the aggregate function is CES. Moreover, as follows from Theorem 7, the curvature of the aggregate function is indeed lower than of its local counterpart (cf. Growiec, 2013).

For the dual problem we assume that

$$F(BK,AL) = (\pi_{0F}(BK)^{\rho} + (1-\pi_{0F})(AL)^{\rho})^{\frac{1}{\rho}}, \qquad \Phi(K,L) = \left(\pi_{0\Phi}K^{\xi} + (1-\pi_{0\Phi})L^{\xi}\right)^{\frac{1}{\xi}}, \tag{69}$$

with $\rho \neq 0$ and $\xi \neq 0$ as well as $\rho < \xi$ which implies $\theta_F(bk) > \theta_\Phi(k)$. From (36) we obtain:

$$\Pi_{0F}(bk)^{\rho} = \Pi_{0\Phi}k^{\xi} \Rightarrow k^*(b) = \left(\frac{\Pi_{0F}}{\Pi_{0\Phi}}\right)^{\frac{1}{\xi-\rho}} b^{\frac{\rho}{\xi-\rho}}. \tag{70}$$

Inserting this choice for all $(B,A) \in \mathbb{R}^2_+$, from (40) we obtain:

$$g(b) = \frac{f(bk^*(b))}{\phi(k^*(b))} = \frac{\left(\pi_{0F}(\frac{\Pi_{0F}}{\Pi_{0\Phi}})^{\frac{\rho}{\xi-\rho}} b^{\frac{\xi\rho}{\xi-\rho}} + (1-\pi_{0F})\right)^{\frac{1}{\rho}}}{\left(\pi_{0\Phi}(\frac{\Pi_{0F}}{\Pi_{0\Phi}})^{\frac{\xi}{\xi-\rho}} b^{\frac{\xi\rho}{\xi-\rho}} + (1-\pi_{0\Phi})\right)^{\frac{1}{\xi}}} = \zeta \cdot (\pi_{0G}b^{\alpha} + (1-\pi_{0G}))^{\frac{1}{\alpha}}, \tag{71}$$

where the elasticity parameter of the resultant technology menu is equal to $\alpha = \frac{\xi\rho}{\xi-\rho}$, consistently with $\xi = \frac{\alpha\rho}{\alpha-\rho}$ from the primal problem. Even more transparently, we obtain the following relationship between the three functions' elasticities of substitution (in line with equation (48)):

$$\frac{1}{\rho} = \frac{1}{\alpha} + \frac{1}{\xi} \qquad \Longleftrightarrow \qquad \frac{\sigma_F}{\sigma_F - 1} = \frac{\sigma_G}{\sigma_G - 1} + \frac{\sigma_\Phi}{\sigma_\Phi - 1}. \tag{72}$$

The multiplicative constant of the derived technology menu is the same ζ as the constant in the primal problem, and π_{0G}, the partial elasticity of the technology menu at the point of normalization, again satisfies (68). All these findings underscore that the CES technology menu and the CES aggregate function are mutually dual.

We also find again that in the special case where $\pi_{0F} = \pi_{0G} = \pi_{0\Phi}$, we have that $k^*(b) = b^{\frac{\rho}{\xi-\rho}}$ and $\zeta = 1$.

The homothetic case. Let us now consider a homothetic local function:

$$F_h(BK, AL) = \pi_{0F}\left(\frac{(BK)^\rho - 1}{\rho}\right) + (1 - \pi_{0F})\left(\frac{(AL)^\rho - 1}{\rho}\right). \tag{73}$$

This monotone transformation of a CES local function (which uses the formula $f_h(x) = \frac{x^\rho - 1}{\rho}$) is often used in the modeling of consumer choices, where it represents additively separable CRRA (constant relative risk aversion) preferences. As argued in the previous sections, the results here are the same as for an untransformed CES local function.

We also note that due to Bergson's theorem (Theorem 1), preferences given by (64) and (73) are in fact the only specifications which are both homothetic and additively separable with respect to BK and AL.

5.3 The Minimum and Maximum Functions

Mutual duality between the technology menu and the aggregate function subject to a minimum (Leontief) local function, along which the factors are perfectly complementary (i.e., *idempotent* duality), has already been identified and thoroughly discussed by Rubinov and Glover (1998); Matveenko (1997, 2010); Matveenko and Matveenko (2015). These studies have also extended this case into n dimensions. For completeness, here we also present the case where the technology menu or the aggregate function is specified as a maximum function.

Leontief local function. The case where the local function is Leontief is very closely related to our Theorems 2–7 but, strictly speaking, cannot be considered as their special case. The reason is that, contrary to our assumptions, the minimum (Leontief) function:

$$F(BK, AL) = \min\{BK, AL\}, \qquad f(bk) = \min\{bk, 1\}, \tag{74}$$

is not differentiable at the point where $BK = AL$. Nevertheless, the results obtained here can still be conveniently characterized as a limiting case of our setup, where the curvature of the local function tends to infinity at the "kink" (i.e., at the ray from the origin satisfying $BK = AL$). Second order conditions are then automatically verified.

Assuming that the curvature of the technology menu is finite, the first order condition for the primal problem implies $bk = 1$ (and thus $b^*(k) = 1/k$) as well as $f(bk) = bk = 1$. Inserting this choice into the local function for all $(K, L) \in \mathbb{R}^2_+$ we obtain:

$$\phi(k) = \frac{1}{g(1/k)}, \tag{75}$$

which is fully in line with (40). Consequently, in line with (47) we obtain that:

$$\pi = \pi_G(1/k) = \pi_\Phi(k), \tag{76}$$

and in line with (48),

$$\frac{1}{1 - \pi - \theta_G(1/k)} = -\frac{1}{1 - \pi - \theta_\Phi(k)}. \tag{77}$$

The solution of the dual problem is fully analogous and implies $k^*(b) = 1/b$ and a technology menu satisfying $g(b) = \frac{1}{\phi(1/b)}$. Equations (76)–(77) are also obtained again, only that one has now to substitute $1/b$ for k.

Furthermore, as demonstrated e.g. by Matveenko and Matveenko (2015), when the local function is Leontief, a Cobb–Douglas technology menu is dual to a Cobb–Douglas aggregate function (and their exponents coincide), in line with (61) and (63). Moreover, a CES technology menu is also dual to a CES aggregate function (and their elasticity parameters are mutually inverse, $\alpha = -\xi$, as in (67) and (71) when taking $\rho \to -\infty$).

Technology menu specified as a maximum function. It is also interesting to consider the primal problem under the extreme assumption that the technology menu is given by a maximum function,

$$G(B, A) = \max\{B, A\}, \qquad g(b) = \max\{b, 1\}. \tag{78}$$

This function, not differentiable at $b = 1$, represents a case where the overall level of technology in the economy is pinned down by the *best* of the available factor-specific technologies. It is thus the limiting case where the trade-off between the quality (unit productivity) of the respective factors, which becomes very small for highly convex technology menus, disappears completely.

The current case can be conveniently characterized as a limiting case of our general setup, where the curvature of the technology menu tends to minus infinity at the "kink" (i.e., the ray from the origin where $B = A$). Second order conditions are then automatically verified.

Assuming that the curvature of the local function is finite, the first order condition for the primal problem implies $b^* = 1$ for all k as well as $f(bk) = f(k)$. Inserting this choice into the local function for all $(K, L) \in \mathbb{R}_+^2$ we obtain:

$$\phi(k) = \frac{f(k)}{g(1)} = f(k), \tag{79}$$

in line with (40). Consequently, in line with (47) we obtain that $\pi_F(k) = \pi_\Phi(k)$, and in line with (48), that $\theta_F(k) = \theta_\Phi(k)$.

Aggregate function specified as a maximum function. Symmetrically, the solution to the dual problem with a maximum aggregate function:

$$\Phi(K, L) = \max\{K, L\}, \qquad \phi(k) = \max\{k, 1\}, \tag{80}$$

is fully analogous and implies $k^* = 1$ for all b and a technology menu satisfying $g(b) = \frac{f(b)}{\phi(1)} = f(b)$, and thus also the partial elasticities and curvatures of the technology menu and the local function are equalized.

Comments. First, the maximum function may look a bit strange as a technology menu and very strange as an aggregate production or utility function. Indeed, we typically expect these functions to be concave, and the maximum function represents extreme convexity. Therefore the economic applications of the above examples, and especially the dual problem, are likely to be limited. They may nevertheless be useful as "cautionary" examples indicating the consequences of assuming that the aggregate function or the technology menu have the same functional form as the local function. Namely, the local and aggregate functions can have the same (non–Cobb–Douglas) form only if the technology menu is a maximum function, i.e., there is no trade-off between the qualities of the respective factors. Analogously, the local function can have the same (non–Cobb–Douglas) form as the technology menu only if the aggregate function is specified as a maximum function.

Second, the maximum case is pathological in the same sense as is the case with a Cobb–Douglas local function – namely that the technology menu and the aggregate function are not mutually dual here because the technology choice is always constant and thus not invertible. Indeed, trying to solve the dual problem with $F = \Phi$,

$$\max_{(K,L)\in\mathbb{R}^2_+} F(BK, AL) \qquad s.t. \qquad F(K, L) = 1, \tag{81}$$

leads to a first order condition of form $\Pi_F(bk) = \Pi_F(k)$ for any given b. This holds either if F is a Cobb–Douglas function, or otherwise only if $b = 1$. The former case has been discussed previously (and flagged as pathological), whereas the latter implies that for $b = 1$ the optimal factor choice is indeterminate, and for $b \neq 1$ there is no interior stationary point. A similar problem is encountered when solving the primal problem for $F = G$. This pathological outcome is a direct consequence of violation of the curvature assumption in Theorem 2 (when solving the primal problem while assuming that F and G have exactly the same functional form) or in Theorem 4 (when making this assumption for F and Φ in the dual problem).

6 The Technology Menu and Distributions of Ideas

Having solved the optimal factor-specific technology choice problem in its generality, let us now comment on one of its particular economic interpretations. Namely, instead of viewing the technology menu as a primitive concept, we shall posit that it has been derived from a certain probabilistic model.

In line with this interpretation, the technology menu $G(B, A)$ has been sometimes viewed as a level curve of a certain two-dimensional complementary cumulative distribution function of (stochastic) factor-specific ideas (unit factor productivities, Jones, 2005; Growiec, 2008a,b). Taking this perspective, our above results pose a range of useful corollaries.

To begin, let us recall that from Sklar's theorem for complementary cumulative distribution functions (ccdfs, see Nelsen, 1999; McNeil and Nešlehová, 2009) we know that any joint distribution can be written as a composition of marginal distributions and a copula:

$$F(x, y) = \mathbb{P}(X > x, Y > y) = C(F_x(x), F_y(y)), \tag{82}$$

where $F_x, F_y : \mathbb{R}_+ \to [0, 1]$ represent the marginal complementary cumulative distribution functions (ccdfs),

$$F_x(x) = \mathbb{P}(X > x), \qquad F_y(y) = \mathbb{P}(Y > y), \tag{83}$$

and $C : [0, 1]^2 \to [0, 1]$ is the copula.

Given this notation, our key finding is that imposing that F should be homothetic heavily narrows the range of distributions we may actually consider for any given copula C. In fact, Bergson's theorem (Theorem 1) can also be applied to copulas.

6.1 Bergson's Theorem for Copulas

Similarly to the original Bergson's theorem (Theorem 1), we find that if F is homothetic and the underlying copula C can be monotonically transformed into an additively separable function, then F – whose level curve is then the technology menu – must be of a very specific, Cobb–Douglas or CES functional form. This form is then translated into very specific requirements imposed on the marginal distributions. For example, as demonstrated in Growiec (2008b), if the marginal distributions are assumed to be independent, homotheticity of the technology menu implies that they must be of the Pareto or Weibull form.

Theorem 8 (Bergson's theorem for copulas) *Let $F_h : \mathbb{R}_+^2 \to [0,1]$ be a homothetic bivariate complementary cumulative distribution function (ccdf) satisfying $F_h(x,y) = C(F_x(x), F_y(y))$, where $F_x, F_y : \mathbb{R} \to [0,1]$ are differentiable marginal ccdfs and $C : [0,1]^2 \to [0,1]$ is a differentiable copula which can be written as additively separable after a monotone transformation:*

$$\exists (f_h : \mathbb{R}_+ \to [0,1],\ F : \mathbb{R}_+^2 \to \mathbb{R}_+) \qquad F_h(x,y) = f_h(F(x,y)), \tag{84}$$

$$\exists (f_s : \mathbb{R} \to [0,1],\ D_u, D_v : [0,1] \to \mathbb{R}) \qquad C(u,v) = f_s(D_u(u) + D_v(v)), \tag{85}$$

where f_h, f_s, D_u, D_v are decreasing differentiable functions, and F is an increasing, differentiable and homogeneous function. Then the technology menu is represented by

$$F(x,y) = c \cdot x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}} \qquad \text{or} \qquad F(x,y) = (\alpha x^\rho + \beta y^\rho)^{\frac{1}{\rho}}, \tag{86}$$

where $\alpha > 0, \beta > 0$; $c_x, c_y \in \mathbb{R}$ are arbitrary constants, $c = \exp\left(\frac{c_x + c_y}{\alpha+\beta}\right)$, and $\rho \neq 0$. Moreover, marginal distributions must satisfy:

$$D_u'(F_x(x))F_x'(x) = \alpha x^{\rho-1}, \qquad D_v'(F_y(y))F_y'(y) = \beta y^{\rho-1}. \tag{87}$$

Proof. We begin by writing down the marginal rate of substitution of the function F_h. On the one hand we have:

$$MRS = -\frac{\frac{\partial F_h}{\partial y}}{\frac{\partial F_h}{\partial x}} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} \equiv -H\left(\frac{x}{y}\right). \tag{88}$$

The function H depends on the x/y ratio only due to the homogeneity of F. On the other hand, however, using the copula representation,

$$MRS = -\frac{\frac{\partial F_h}{\partial y}}{\frac{\partial F_h}{\partial x}} = -\frac{\frac{\partial C}{\partial v} F_y'(y)}{\frac{\partial C}{\partial u} F_x'(x)} = -\frac{D_v'(F_y(y))F_y'(y)}{D_u'(F_x(x))F_x'(x)} \equiv -\frac{H_y(y)}{H_x(x)}. \tag{89}$$

Therefore $H(\frac{x}{y}) = \frac{H_y(y)}{H_x(x)}$ for all $x, y \in \mathbb{R}_+$. Differentiating both sides of this functional equality with respect to x and y and eliminating $H'(\frac{x}{y})$, we obtain:

$$\frac{H_x'(x)x}{H_x(x)} = \frac{H_y'(y)y}{H_y(y)}, \qquad \text{for all } x, y \in \mathbb{R}. \tag{90}$$

Therefore both sides of (90) must be constant. We denote this constant $\rho - 1$. Integrating, we obtain:

$$H_x(x) = \alpha x^{\rho-1}, \qquad H_y(y) = \beta y^{\rho-1}, \tag{91}$$

for some $\alpha, \beta \in \mathbb{R}$. Substituting for $H_x(x)$ and $H_y(y)$ and observing the signs of respective derivatives yields (87).

This also implies that $H(\frac{x}{y}) = \frac{\beta}{\alpha}\left(\frac{x}{y}\right)^{1-\rho}$, and thus the homogeneous function $F(x, y)$ must take either the Cobb–Douglas or the CES form (86) (cf., Arrow, Chenery, Minhas, and Solow, 1961). ■

6.2 Application to Archimedean Copulas

Theorem 8 has quite broad applicability. It affects not only the case where both idea distributions are independent, but also the case where they are mutually dependent and their dependence is modeled by some representative of the broad and widely applied Archimedean class of copulas, e.g. Clayton, Gumbel, Ali-Mikhail-Haq, Frank, Joe, etc.

Indeed, by definition each bivariate Archimedean copula can be written as (McNeil and Nešlehová, 2009):

$$C(u, v) = \psi(\psi^{-1}(u) + \psi^{-1}(v)), \tag{92}$$

where $\psi : \mathbb{R}_+ \to [0, 1]$ is a decreasing, continuous function satisfying $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$. The function ψ is called the *Archimedean generator*.

Hence, it suffices to take $f_s = \psi$ and $D_u = D_v = \psi^{-1}$ in the assumptions of Theorem 8 to observe that in fact all Archimedean copulas are subject to this theorem. Thus, when we assume homotheticity of the joint idea distribution and model dependence of its marginal distributions by the means of a specific Archimedean copula[14], the technology menu must take the Cobb–Douglas or CES form, implying that the shapes of the marginal distributions must satisfy a very specific parametric condition which is unique for the given copula.

More precisely, for Archimedean copulas we obtain from (87):

$$\frac{\partial}{\partial x}(\psi^{-1}(F_x(x))) = \alpha x^{\rho-1}, \qquad \frac{\partial}{\partial y}(\psi^{-1}(F_y(y))) = \beta y^{\rho-1}, \qquad \alpha > 0, \beta > 0, \rho \in \mathbb{R}. \tag{93}$$

Integrating, we obtain that $F_x(x)$ and $F_y(y)$ must necessarily follow the formula:

$$F_x(x) = \psi\left(c_x + \alpha \ln x\right) \text{ if } \rho = 0, \qquad F_y(y) = \psi\left(c_y + \beta \ln y\right) \text{ if } \rho = 0, \tag{94}$$

$$F_x(x) = \psi\left(c_x + \frac{\alpha}{\rho} x^\rho\right) \text{ if } \rho \neq 0, \qquad F_y(y) = \psi\left(c_y + \frac{\beta}{\rho} y^\rho\right) \text{ if } \rho \neq 0, \tag{95}$$

where c_x, c_y are arbitrary constants of integration.

Owing to the properties of ψ, it is easily verified that $F_x(x)$ and $F_y(y)$ are indeed decreasing functions. Moreover, if $\rho \geq 0$ then $\lim_{x\to\infty} F_x(x) = \lim_{y\to\infty} F_y(y) = 0$. Other properties depend on the exact choice of the generator ψ and parameters. In particular, for some parametrizations the supports of random variables X and Y may be limited. In such a case, F_x or F_y should be set identically to zero for arguments exceeding the upper bound of the support and to unity for arguments below the lower bound of the support. Then the technology menu should also be defined only on this particular limited support.

Below we briefly review a few of the most common Archimedean copulas. In each case, we derive the exact functional form that the marginal ccdfs must follow in order to be consistent with

[14] For example, Growiec (2008a) modeled the dependence of marginal idea distributions with a Clayton copula. His study, however, did not assume homotheticity (apart from a few special cases).

homotheticity of the technology menu.

Independent marginal distributions. The independence copula takes the form $C(u, v) = uv$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z) = \psi(z) = e^{-z}, D_u(u) = \psi^{-1}(u) = -\ln u, D_v(v) = \psi^{-1}(v) = -\ln v$. We then obtain:

$$F_x(x) = e^{-c_x} x^{-\alpha} \text{ if } \rho = 0, \qquad F_y(y) = e^{-c_y} y^{-\beta} \text{ if } \rho = 0, \tag{96}$$

$$F_x(x) = e^{-c_x} e^{-\frac{\alpha}{\rho} x^\rho} \text{ if } \rho \neq 0, \qquad F_y(y) = e^{-c_y} e^{-\frac{\beta}{\rho} y^\rho} \text{ if } \rho \neq 0. \tag{97}$$

This means that, as found by Growiec (2008b), if the marginal distributions are independent, homotheticity of the technology menu implies that these distributions must take either the Pareto (96) or the Weibull form ((97) with $\rho > 0$). In the latter case, both marginal distributions must have equal exponents (i.e., shape parameters).

Clayton copula. Clayton copula takes the form $C(u, v) = (\max\{0, u^\delta + v^\delta - 1\})^{\frac{1}{\delta}}$, with $\delta \leq 1$ and $\delta \neq 0$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z) = \psi(z) = (1 - \delta z)^{\frac{1}{\delta}}$ as well as $D_u(u) = \psi^{-1}(u) = -\frac{1}{\delta}\left(u^\delta - 1\right), D_v(v) = \psi^{-1}(v) = -\frac{1}{\delta}\left(v^\delta - 1\right)$. We then obtain:

$$F_x(x) = (c_x - \alpha\delta \ln x)^{\frac{1}{\delta}} \text{ if } \rho = 0, \qquad F_y(y) = (c_y - \beta\delta \ln y)^{\frac{1}{\delta}} \text{ if } \rho = 0, \tag{98}$$

$$F_x(x) = \left(c_x - \frac{\alpha\delta}{\rho} x^\rho\right)^{\frac{1}{\delta}} \text{ if } \rho \neq 0, \qquad F_y(y) = \left(c_y - \frac{\beta\delta}{\rho} y^\rho\right)^{\frac{1}{\delta}} \text{ if } \rho \neq 0, \tag{99}$$

Of particular interest is the case (99) with $c_x = c_y = 0$ as well as $\delta\rho < 0$. It implies that x and y are Pareto distributed with equal exponents (shape parameters) $\frac{\rho}{\delta}$ (Growiec, 2008a).

Gumbel copula. Gumbel copula takes the form $C(u, v) = \exp\left(-((-\ln u)^\delta + (-\ln v)^\delta)^{\frac{1}{\delta}}\right)$, with $\delta \geq 1$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z) = \psi(z) = e^{-z^{\frac{1}{\delta}}}$ as well as $D_u(u) = \psi^{-1}(u) = (-\ln u)^\delta, D_v(v) = \psi^{-1}(v) = (-\ln v)^\delta$. We then obtain:

$$F_x(x) = e^{-(c_x + \alpha \ln x)^{\frac{1}{\delta}}} \text{ if } \rho = 0, \qquad F_y(y) = e^{-(c_y + \beta \ln y)^{\frac{1}{\delta}}} \text{ if } \rho = 0, \tag{100}$$

$$F_x(x) = e^{-\left(c_x + \frac{\alpha}{\rho} x^\rho\right)^{\frac{1}{\delta}}} \text{ if } \rho \neq 0, \qquad F_y(y) = e^{-\left(c_y + \frac{\beta}{\rho} y^\rho\right)^{\frac{1}{\delta}}} \text{ if } \rho \neq 0. \tag{101}$$

Of particular interest is the case (101) with $c_x = c_y = 0$ as well as $\delta\rho > 0$. It implies that x and y are Weibull distributed with equal exponents (shape parameters) $\frac{\rho}{\delta}$.

Ali-Mikhail-Haq copula. Ali-Mikhail-Haq copula takes the form $C(u, v) = \frac{uv}{1-\delta(1-u)(1-v)}$, with $\delta \in [-1, 1)$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z) = \psi(z) = \frac{1-\delta}{e^z - \delta}$ as well as $D_u(u) = \psi^{-1}(u) = \ln\left(\frac{1-\delta(1-u)}{u}\right), D_v(v) = \psi^{-1}(v) = \ln\left(\frac{1-\delta(1-v)}{v}\right)$. We then obtain:

$$F_x(x) = \frac{1-\delta}{x^\alpha e^{c_x} - \delta} \text{ if } \rho = 0, \qquad F_y(y) = \frac{1-\delta}{y^\beta e^{c_y} - \delta} \text{ if } \rho = 0, \tag{102}$$

$$F_x(x) = \frac{1-\delta}{e^{\frac{\alpha}{\rho} x^\rho} e^{c_x} - \delta} \text{ if } \rho \neq 0, \qquad F_y(y) = \frac{1-\delta}{e^{\frac{\beta}{\rho} y^\rho} e^{c_y} - \delta} \text{ if } \rho \neq 0. \tag{103}$$

In sum, when the technology menu $G(B, A)$ is viewed as a level curve of a certain two-dimensional complementary cumulative distribution function of (stochastic) factor-specific ideas, then the probabilistic structure of the underlying model imposes severe restrictions on the range of available functional forms. In particular, if the marginal idea distributions are either independent, or dependent according to some Archimedean copula, then – coupled with the homotheticity assumption – this implies that the technology menu must take the Cobb–Douglas or CES form.

7 Conclusion

In this article, we have provided a detailed treatment of a static, two-dimensional problem of factor-specific technology choice. At the core of this problem there is a local function F, along which the factors are multiplied by their respective unit productivities, drawn from a certain technology menu G. We have derived the optimal technology choices in such a setup and constructed the aggregate function Φ as an envelope of the local functions. We have also solved a symmetric dual problem where Φ is given, and G – sought.

It turns out that the properties of this optimization problem can be characterized with the use of a generalized notion of duality ("F-duality"). In the optimum, partial elasticities of F, G and Φ are all equal, and there exists a clear-cut and economically interpretable relationship between their curvatures.

Our results are marked by their generality and broad applicability. At the same time, however, they also underscore how restrictive the assumptions of homogeneity (constant returns to scale) and homotheticity can be. Crucially, by the virtue of Bergson's theorem (Burk, 1936) homotheticity, when coupled with additive separability, implies the Cobb–Douglas or CES functional form. As we have demonstrated, this result has most bite when one envisages the technology menu as a level curve of a certain bivariate distribution of ideas (Jones, 2005; Growiec, 2008a).

The current study can be extended in a variety of directions as well as applied in a variety of contexts. The most needed theoretical extensions include accomodating non-homothetic local functions and technology menus as well as increasing the dimensionality of the problem by considering more than two factors. These tasks have already been accomplished for special cases such as Cobb–Douglas, CES or Leontief functions. To be addressed in their generality, however, they require the modeler to give up additive separability – a particular inconvenience in higher dimensions – and to make certain decisions with regard to the preferred measures of curvature in higher dimensions, which may just as well mean an opening of Pandora's box.

The scope for applications of the discussed framework is even broader. Firstly, while thus far optimal factor-specific technology choice has been studied predominantly in the context of growth theory, it may just as well be incorporated in models of, e.g., industrial organization, international trade, natural resources, sectoral change, consumption patterns, or social welfare. Secondly, the static technology choice problem studied here could be given a dynamic edge by assuming that the technology is fixed in the short run but not in the long run, and thus the local function represents the short-run technology whereas the aggregate function holds only in the long run. León-Ledesma and Satchi (2016) are the first to formalize this idea, constructing a model where capital and labor are gross complements in the short run but over the long run the technology is Cobb–Douglas. In this way they circumvent the Steady State Growth Theorem (Uzawa, 1961) and reconcile the long-run balanced growth requirement with the mounting empirical evidence of gross complementarity of both factors and non-neutral technical change. Their brilliant idea can clearly be taken further, with a wide range of potential extensions and applications.

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