Real World Scenarios for Interest Rates based on the LIBOR Market Model

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Abstract
In this article, we present a methodology to simulate the evolution of interest rates under real world probability measure. More precisely, using the multidimensional LIBOR market model and a specification of the market price of risk vector process, we explain how to perform simulations of the real world forward rates in future, using the Euler scheme with Predictor-Corrector. The proposed methodology allows the presence of negative interest rates as currently observed in the markets.

Keywords: Real World Model, Scenario Simulation, Interest Rate, LFM, Market Price of Risk

1. INTRODUCTION
The increase of regulatory and internal demands on risk assessment and management of assets and liabilities within banks and insurance companies led to the need of a better understanding of the uncertainty in market risk factors, particularly in future scenarios of interest rates.

For pricing financial products whose value today depends on future realizations of a certain risk factor, one can rely on risk neutral models and prices are obtained as the discounted value of expected future payoffs under the standard hypothesis on frictionless and complete markets. However, when the objective is the simulation of real future values of these underlying factors and products, as is the case in the assessment of investment strategies in interest-rate sensitive portfolios for Asset-Liability Management studies and calculations of Economic Capital for Solvency II, the risk neutral probabilities do not represent real probabilities, as the drift of stock prices is assumed to be the risk free rate and that the forward rates are unbiased predictors of future rates, which is not realistic because that would imply that investors require no compensation for the risk of unpredicted changes in the future.

In order to simulate future trajectories of financial factors, i.e. to simulate how the world will look like in the future, one should use a probability measure that reflects the fact that investors demand a risk premium to hold risky assets [1].

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One of the main problems in interest rate modeling is the choice of the appropriate interest rates. One can choose to model the instantaneous spot rate using short rate models (see e.g. [2], [3]) and use their simplistic structure to take advantage of existing closed pricing formulas. However, the main drawback of short rate models derives from the fact that the instantaneous spot rates are not directly observed in the markets.

In this article, we choose to model a set of key forward LIBOR rates using the LIBOR Market Model [4], which can be directly observed in the market, guarantee no arbitrage opportunities in interest rate markets and provide more flexibility to capture all possible curve movements.

In Section II we introduce the LIBOR market model and its risk neutral dynamics. In Section III we define the market price of risk process, describe the interest rates dynamics under the real world measure and present the parameters estimation and simulation methodologies we follow. In the last section we present the results of the projected rates one year ahead from the last data point using historical data of AAA-rated euro bonds from September 2004 until July 2015.

2. The Risk Neutral Model for LIBOR Rates

LIBOR market models for interest rates have become very popular mainly due to the agreement between such models and market formulas for pricing two basic derivative products: caps and swaps. More precisely, the lognormal forward-LIBOR model (LFM) as termed in [5]) allows to price caps with Black’s cap formula used in the cap market. For a detailed guide into interest rate models, see [5].

In this framework, we consider the tenor structure $\mathcal{T} = \{T_0, T_1, ..., T_{N+1}\}$ with $T_0 = 0$, and where $T_j < T_k$ for $0 \leq j < k \leq N + 1$ and define the corresponding accruals as $\delta_j = T_{j+1} - T_j$, $0 \leq j \leq N$. For $j = 0, 1, ..., N + 1$, let us denote by $B_j(t)$ the price at time $t$ of a zero-coupon bond that matures at the tenor date $T_j$ with $T_j \geq t$. Moreover, for $j = 0, 1, ..., N$, let us define by $F_j(t) = F(t, T_j, T_{j+1})$ the value at time $t$ of the forward rate for the period $[T_j, T_{j+1}]$.

The forward LIBOR rates can be obtained in terms of the bond prices by using the following relation:

$$F_j(t) = \frac{B_j(t) - B_{j+1}(t)}{\delta_j B_{j+1}(t)}. \quad (1)$$

We define as numeraire asset the discretely balanced bank account, the value of which at time $t$ is given by:

$$B_d(t) = \prod_{j=0}^{m(t)-1} (1 + \delta_j F_j(T_{j-1})) B(t, T_{m(t)-1}), \quad (2)$$

where $m(t)$ is the notation for the next tenor date after time $t$, i.e., $m(t) = T_j$ if $T_{j-1} \leq t < T_j$. The interpretation of $B_d(t)$ is that of the value of a portfolio that starts with one unit of currency at time 0 and reinvesting at each tenor date in zero-coupon bonds for the next tenor. It can be thought as the discrete version of the continuous compounded bank account.
The measure associated with the numeraire $B_d$ is called the spot measure, under which the dynamics of the LIBOR rates is the following:

$$\frac{dF_j(t)}{F_j(t)} = \mu_j(t)dt + \epsilon_j(t) \cdot dW(t),$$

where:

$$\mu_j(t) = \sum_{i=m(t)}^{j} \frac{\delta \epsilon_j(t) \cdot \epsilon_i(t) F_i(t)}{1 + \delta F_i(t)},$$

and $\epsilon_j(t) = \{\epsilon^1_j(t), ..., \epsilon^d_j(t)\}$ is a vector of volatility functions, and $W(t) = \{W^1(t), ..., W^d(t)\}$ denotes a multidimensional Brownian motion.

We consider that the volatility functions are deterministic functions which depend only on the whole reset periods between time $t$ and the maturity of the rate, i.e:

$$\epsilon^p_j(t) = \lambda^p_j - m(t),$$

and we write the volatility functions in terms of an orthonormal basis $\Lambda^p_i$:

$$\lambda^p_j = \sigma^p \Lambda^p_j.$$  

3. The Real World Model for LIBOR Rates

As already mentioned, the real world dynamics of interest rates will be useful to predict future evolutions in the markets. Moreover, in order to consistently perform the simulations, we need to understand the past and incorporate that information in the model in which the decision making process will be based. So, instead of calibrating the model to current prices we use historical estimation of the parameters of the model. In this section we present the market price of risk process, its incorporation in the risk neutral LIBOR market model, the estimation methodology and how to obtain the scenarios.

From the adopted risk neutral model and using Girsanov’s Theorem, we can write the dynamics of the LIBOR rates under the real world measure using the following relation between the $d$-dimensional Brownian motion under the real world measure, $W^P$, and the $d$-dimensional Brownian motion under the spot measure $W^Q$:

$$dW^P(t) = dW^Q(t) + \theta(t)dt,$$

where $\theta(t)$ is the $d$-dimensional market price of risk process.

Let us consider the process of market price of risk $\theta(t)$ as the excess return for taking a unit amount of interest rate risk. Wilmott and Ahmad [6] argue that since investors are not always rational, this process should not be modeled as a constant (nor even as a piecewise constant function). Also, for a fixed time, the market price of risk must be common to every interest rate derivative (when the underlying interest rates are common in both instruments).

$$\sum_{i=0}^{N-1} \Lambda^p_i \Lambda^q_i = \delta^{pq}$$ where $N$ can be interpreted to be the number of maturities observable on the forward rate curve.
The specification of the market price of risk process adopted in [7] takes into account these two properties. So, the $p$ component of the market price of risk has the following specification:

$$\theta^p(t) = \frac{a^p}{\sigma^p} \left( b^p - \sum_{i=m(t)}^{N+m(t)-1} \Lambda_i^p F_i(t) \right),$$  \hspace{1cm} (5)

where $a^p$, $\sigma^p$ e $b^p$ are parameters to be estimated from historical data and the vectors $\Lambda_i$ are the coefficients obtained from the principal component analysis on the interest rates covariance matrix.

Under this specification of the market price of risk process, we assume that the market price of risk is a linear function of the observed forward rates and introduce a mean reversion dynamics in the forward rates, where the parameters $b^p$ and $a^p$ represent the long run mean and speed of reversion of the $p$th factor, respectively.

We assume the following real world multidimensional dynamics for the forward LIBOR rates:

$$\frac{dF_j(t)}{F_j(t)} = (\mu_j(t) + \theta(t) \cdot \epsilon_j(t)) dt + \epsilon_j(t) \cdot dW(t).$$  \hspace{1cm} (6)

Since we only have a finite number of observed rates, we need to extrapolate the interest rates curve. For this purpose, we use the following recursion formula for future and unobservered rates:

$$F_{i+N}(t) = a \left( (1+\beta)F_{i+N-1}(t) - \beta F_{i+N-2}(t) \right) + (1-a)c + \sigma \epsilon, \hspace{0.5cm} i = 1, 2, \ldots,$$  \hspace{1cm} (7)

where the parameters $a$ e $\beta$ reflect, respectively, the dependence on the long term interest rate $c$ and how much dependence on the slope of the yield curve the new rate has. Moreover, $\epsilon$ is a normal random variate and $\sigma$ its standard error. With this specification, $a$, $\beta$ and $\sigma$ can be chosen by using the risk manager’s view of the future and $c$ can reflect the market expectations on the long term evolution of the interest rates.

4. **Parameter Estimation**

This section details the method for calibrating the real world model to historical data. We mainly follow the methodology introduced in [7] with the appropriate modifications, since we are modeling changes in the rates instead of changes in the logarithm of rates in order to allow for negative interest rates.

First, for simplicity sake, we consider a common constant accrual, i.e. $\delta = \delta_j = T_{j+1} - T_j$, for $j = 1, \ldots, N$, and adopt the following notation for the constant maturity forward rates:

$$L_j(t) = F(t, t+\delta_j, t+(j+1)\delta), \hspace{0.5cm} j = 0, \ldots, N-1.$$  

More precisely, $L_j(t)$ denotes the value at time $t$ of the forward rate for the period $[t+\delta_j, t+(j+1)]$.

Next, we consider the covariance matrix $\Sigma$ of the historical changes in the forward rates and, using Principal Components Analysis, we recover the $\Lambda_i^p$’s and $\sigma_i^p$’s as the eigenvectors and
eigenvalues of $\Sigma$, respectively. Also in this step, we define the dimension of the volatility functions and the number of components of the market price of risk process. Empirically, changes in interest rates can be largely explained by a small number of factors: level, slope, bow, and higher order perturbations.

By considering a small time step, $\Delta t$, between observations and using the Euler-Maruyama scheme, we approximate the forward rates as follows:

$$L_j(t + \Delta t) \approx L_j(t) + (\lambda_j \cdot \theta(t) + \bar{\mu}_j(t)) L_j(t) \Delta t + L_j(t) \lambda_j \cdot \Delta W,$$

where

$$\bar{\mu}_j(t) = \sum_{k=0}^{j} \frac{\delta \lambda_j \lambda_k L_j(t)}{1 + \delta L_k(t)},$$

$$\bar{L}_j(t) = L_j(t) + \bar{\mu}_j(t) L_j(t) \Delta t$$

Thus, by multiplying (11) by $\Lambda^p_j$ and summing over $j$, we obtain

$$f^p(t + \Delta t) = \bar{f}^p(t) + a^p(b^p - f^p(t)) S(t) \Delta t + S(t) \sigma^p \Delta W^p$$

for $p = 1, 2, \ldots d$, (12)

where we have introduced

$$f^p(t) = \sum_{j=0}^{N-1} \Lambda^p_j L_j(t),$$

$$\bar{f}^p(t) = \sum_{j=0}^{N-1} \Lambda^p_j \bar{L}_j(t)$$

and

$$S(t) = \sum_{j=0}^{N-1} L_j(t).$$

In order to estimate the parameters $a^p$ and $b^p$, first from the historical data we define the following series of observations:

$$Y^p(t) = \frac{f^p(t + \Delta t) - \bar{f}^p(t)}{S(t)}$$

and

$$X^p(t) = f^p(t).$$

Next, we estimate the $p$ regression models obtained from equation (12):

$$Y^p(t) = c^p + m^p X^p(t) + e^p(t),$$

where

$$e^p(t) = \sigma^p \Delta W^p$$

$$m^p = -a^p \Delta t$$

$$c^p = a^p b^p \Delta t$$

so that we can recover $a^p$ and $b^p$ from $m^p$ and $c^p$. 
5. **Real World Scenarios Simulation**

After obtaining all the market price of risk coefficients, we take the discretized model \[^{[8]}\] and the predictor-corrector method for our simulations. In this case, the SDE describing the evolution of the forward rates \[^{[6]}\] involves a state dependent drift, which implies that there is no analytic solution to the SDE so that it must be numerically approximated.

Many authors proposed different approximations schemes to simulate the risk neutral dynamics of forward LIBOR rates as the predictor-corrector approximation \[^{[8]}\], the drift-free simulation method \[^{[9]}\] and also the parameterized drift-free simulation (PDFS) \[^{[10]}\].

Here, since our simulations are done under the real world measure, we use the predictor-corrector method. The idea is to first evolve forward rates pretending that all state variables in the drift are constant (frozen at the previous time step), recompute the drift at the evolved time and average the two drifts. Next, we recompute the forward rates using this averaged drift and the same random numbers.

At step one we compute drifts using the observed rates:

\[
\mu_{j}(t) = \sum_{k=0}^{j} \frac{\lambda_{j} \lambda_{k} L_{k}(t)}{1 + \delta L_{k}(t)}
\]

(18)

Next, we repeat the previous step using the \(L_{j}^{\text{aux}}\)'s instead of \(L_{j}\)'s to retrieve \(\mu^{\ast}_{j}(t)\) and obtain the final approximation of the rates as:

\[
L_{j}(t + \Delta t) = L_{j}(t) + \left( \lambda_{j} \cdot \frac{\theta(t) + \theta^{\ast}(t)}{2} + \frac{\mu_{j}(t) + \mu^{\ast}_{j}(t)}{2} \right) L_{j}(t) \Delta t + L_{j}(t) \lambda_{j} \cdot \Delta W,
\]

(19)

where

\[
\theta^{q}(t) = \frac{a_{q}}{\sigma_{q}} \left( b^{q} - \sum_{j=0}^{N-1} \Lambda^{q}_{j} L_{j}(t) \right) \text{ for } q = 1, \ldots, d
\]

(20)

and

\[
\theta^{\ast q}(t) = \frac{a_{q}}{\sigma_{q}} \left( b^{q} - \sum_{j=0}^{N-1} \Lambda^{q}_{j} L_{j}^{\text{aux}}(t) \right) \text{ for } q = 1, \ldots, d.
\]

(21)

5.1. Results

In this section we present the estimation and simulation results of the real world LIBOR market model. We also give a simple application of Value at Risk calculation using the generated scenarios. We use a four factor version of the model.

The parameter estimates are obtained from the historical observations of European AAA-government bonds prices ranging from 30/09/2004 up to 31/07/2015, which are available in the European Central Bank database. **Table 1** presents the forward rate curve observed in July 31, 2015. At that time, negative forward rates of one year tenor for the maturities of one and two years were prevailing.
<table>
<thead>
<tr>
<th>Maturity</th>
<th>Forward Rate</th>
<th>Maturity</th>
<th>Forward Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0,221%</td>
<td>16</td>
<td>1,975%</td>
</tr>
<tr>
<td>2</td>
<td>-0,003%</td>
<td>17</td>
<td>1,929%</td>
</tr>
<tr>
<td>3</td>
<td>0,289%</td>
<td>18</td>
<td>1,877%</td>
</tr>
<tr>
<td>4</td>
<td>0,605%</td>
<td>19</td>
<td>1,819%</td>
</tr>
<tr>
<td>5</td>
<td>0,991%</td>
<td>20</td>
<td>1,759%</td>
</tr>
<tr>
<td>6</td>
<td>1,189%</td>
<td>21</td>
<td>1,698%</td>
</tr>
<tr>
<td>7</td>
<td>1,429%</td>
<td>22</td>
<td>1,637%</td>
</tr>
<tr>
<td>8</td>
<td>1,625%</td>
<td>23</td>
<td>1,578%</td>
</tr>
<tr>
<td>9</td>
<td>1,799%</td>
<td>24</td>
<td>1,520%</td>
</tr>
<tr>
<td>10</td>
<td>1,893%</td>
<td>25</td>
<td>1,465%</td>
</tr>
<tr>
<td>11</td>
<td>1,970%</td>
<td>26</td>
<td>1,413%</td>
</tr>
<tr>
<td>12</td>
<td>2,016%</td>
<td>27</td>
<td>1,364%</td>
</tr>
<tr>
<td>13</td>
<td>2,034%</td>
<td>28</td>
<td>1,319%</td>
</tr>
<tr>
<td>14</td>
<td>2,031%</td>
<td>29</td>
<td>1,277%</td>
</tr>
<tr>
<td>15</td>
<td>2,010%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: One year Forward rates observed in July 31, 2015

By using the historical forward rates, we compute the covariance matrix $\Sigma$ between monthly changes in the one year forward rates of term $i$ and term $j$ ($i, j = 1, \ldots, 29$). The first four principal components corresponding to the decomposition of the covariance matrix are shown in Figure 1. In this figure we can identify the level, slope, bow, and higher order perturbations factors, as usually in yield curve studies. The first four principal components explain a 99.4% of the covariance of the data.

![Figure 1: The first four principal components obtained from a Principal Components Analysis of monthly observed European forward rates](image)
The parameters of the market price of risk process are shown in Table 2. The results show low mean reversion rate parameters, meaning that any disturbance on the factors has a long term effect on the future rates. This can be explained by noting that we in include in our sample a crisis period which was followed by a slow economic recovery. Also, the European Central Bank has had a strong influence in current levels of the European bond interest rates specifically with the Expanded Asset Purchase Programme and Covered Bond Purchase Programmes.

<table>
<thead>
<tr>
<th>Factor p</th>
<th>$a^p$</th>
<th>$b^p$</th>
<th>$c^p$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>-0.1886</td>
<td>0.3147</td>
<td>0.7409</td>
</tr>
<tr>
<td>2</td>
<td>0.0553</td>
<td>3.0610</td>
<td>0.1696</td>
</tr>
<tr>
<td>3</td>
<td>0.0045</td>
<td>73.3016</td>
<td>0.0554</td>
</tr>
<tr>
<td>4</td>
<td>-0.0051</td>
<td>-77.7292</td>
<td>0.0282</td>
</tr>
</tbody>
</table>

Table 2: Estimated Market Price of Risk Parameters

In Figure 2 we exhibit 1000 paths for the forward rate curve in the 1st year of simulations; in Figure 3 we can compare the simulated curves with the last observed forward curve, and finally in Figure 4 we present the corresponding zero coupon bond prices simulation results. The expected value and 90% confidence interval for each zero coupon bond price with maturities 1 to 29 years were computed are shown in Table 3.
Figure 4: 1000 Simulations of the zero coupon bond prices at the end of the first projection year

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean</th>
<th>Conf Interval</th>
<th>Maturity</th>
<th>Mean</th>
<th>Conf Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.002288</td>
<td>[1.002286, 1.002289]</td>
<td>16</td>
<td>0.808507</td>
<td>[0.808274, 0.808741]</td>
</tr>
<tr>
<td>2</td>
<td>1.002316</td>
<td>[1.002315, 1.002318]</td>
<td>17</td>
<td>0.793162</td>
<td>[0.792910, 0.793414]</td>
</tr>
<tr>
<td>3</td>
<td>0.999394</td>
<td>[0.999392, 0.999399]</td>
<td>18</td>
<td>0.778493</td>
<td>[0.778224, 0.778762]</td>
</tr>
<tr>
<td>4</td>
<td>0.993372</td>
<td>[0.993362, 0.993382]</td>
<td>19</td>
<td>0.764512</td>
<td>[0.764225, 0.764798]</td>
</tr>
<tr>
<td>5</td>
<td>0.984429</td>
<td>[0.984407, 0.984451]</td>
<td>20</td>
<td>0.751212</td>
<td>[0.750909, 0.751516]</td>
</tr>
<tr>
<td>6</td>
<td>0.972930</td>
<td>[0.972893, 0.972967]</td>
<td>21</td>
<td>0.738578</td>
<td>[0.738258, 0.738898]</td>
</tr>
<tr>
<td>7</td>
<td>0.959330</td>
<td>[0.959275, 0.959385]</td>
<td>22</td>
<td>0.726582</td>
<td>[0.726246, 0.726918]</td>
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<tr>
<td>8</td>
<td>0.944109</td>
<td>[0.944034, 0.944183]</td>
<td>23</td>
<td>0.715192</td>
<td>[0.714840, 0.715545]</td>
</tr>
<tr>
<td>9</td>
<td>0.927728</td>
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<td>10</td>
<td>0.910608</td>
<td>[0.910492, 0.910724]</td>
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<td>11</td>
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<td>13</td>
<td>0.858126</td>
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<td>14</td>
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<tr>
<td>15</td>
<td>0.824496</td>
<td>[0.824281, 0.824711]</td>
<td>30</td>
<td>0.650000</td>
<td>[0.649353, 0.650647]</td>
</tr>
</tbody>
</table>

Table 3: Expected value and 90% confidence interval of zero coupon bond prices
5.2. Example: Value at Risk of Bond Portfolio

Consider an investor that holds a portfolio composed of AAA-rated European bonds of maturities ranging from 1 to 29 years. The quantity held by the investor in bonds with maturity \( j \) is given by \( w_j \).

Consider the following portfolio:

- \( w_1 = w_{10} = 100 \)
- \( w_2 = w_{21} = 20 \)
- \( w_3 = w_{15} = 20 \)
- \( w_5 = w_{13} = w_{17} = w_{29} = 25 \)
- \( w_7 = 50 \)
- \( w_{12} = w_{23} = 15 \)
- \( w_{14} = 200 \)
- \( w_{16} = w_{22} = w_{24} = w_{26} = 10 \)
- \( w_{18} = 30 \)
- \( w_j = 0 \) for \( j = 4, 6, 8, 9, 11, 15, 19, 20, 25, 27, 28 \)

The value of this portfolio in July 31, 2015 was 757.5748 euros.

In Figure 4 the histogram of Profit and Losses simulated as in one year from July 31, 2015 is shown.

![Figure 5: Histogram of one year Profits and Losses simulations](image)

The one year Value at Risk at the \((1 - \alpha)\) level of a portfolio is defined as the \(\alpha\)% quantile of the distribution of profit and losses and it represents the minimum loss that occurs during a year with probability of \(\alpha\).

In Table 4 we present the results of Value at Risk for three different confidence values.

<table>
<thead>
<tr>
<th>(\text{VaR}_{90%})</th>
<th>(\text{VaR}_{95%})</th>
<th>(\text{VaR}_{99%})</th>
<th>(\text{VaR}_{99.5%})</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.7374</td>
<td>11.7282</td>
<td>16.8677</td>
<td>18.9407</td>
</tr>
</tbody>
</table>

Table 4: One-year VaR results
6. Conclusion

We have presented a model specification for real world interest rate simulations by using the Libor Market Model and a market price of risk process specification with parameter estimates obtained by principal component decomposition. The proposed model allows for negative interest rates, as the ones observed in current markets and maintaining realistic levels and shapes for the interest rate curves.

The model can be used in many applications, for example as a tool in Asset-Liability Management and for internal model calculations of the interest rate module of Solvency Capital Requirement, as an alternative to the standard formula proposed by regulators.

As future work we plan to incorporate credit risk and simulate paths for defaultable bonds under the real world LIBOR Market Model.

References


