Term Structure of Interest Rates:
Macro-Finance Approach

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Abstract

The paper focuses on derivation of macro-finance model for analysis of yield curve and its dynamics using macroeconomic factors. Underlying model is based on basic Dynamic Stochastic General Equilibrium (DSGE) approach that stems from Real Business Cycle theory and New Keynesian Macroeconomics. The model includes four main building blocks: households, firms, government, and central bank. Log-linearized solution of the model serves as an input for derivation of yield curve and its main determinants – pricing kernel, price of risk, and affine term structure of interest rates – based on no-arbitrage assumption. This study shows a possible way of consistent derivation of structural macro-finance model, with reasonable computational burden that allows for time varying term premia. A simple VAR model, widely used in macro-finance literature, serves as a benchmark. The paper also presents a brief comparison and shows an ability of both models to fit an average yield curve observed from the data. Lastly, the importance of term structure analysis is demonstrated using case of Central Bank deciding about policy rate and Government conducting debt management.

Keywords: New Keynesian macroeconomics, dynamic stochastic general equilibrium model, fiscal policy, solution of a DSGE model, impulse response functions.


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1 Non-technical summary

The paper focus on macro-finance models that try to analyze and explain yield curve and its dynamics using macroeconomic variables as underlying factors. These models represent a growing part of financial economics having implications for both, macro and financial models.

Currently widespread Dynamic Stochastic General Equilibrium (DSGE) models are predominantly used for carrying out simulations of various economic policies and/or forecasting. Their main builders and users are naturally Central Banks and some Ministries of Finances and Treasuries, but they are also of interest for researchers. The purpose is to give a notion about main links in an economy, about impacts that various policy measures might have on real variables or possibly forecast future development. These models thus include links to analyze impacts between variables such as private consumption, inflation, unemployment, government spending and interest rate among others. For the last, usually one period short term rate is used, i.e. 3-months rate in quarterly models. This interest rate then has an impact e.g. on decision of households about consumption, on firms costs and investment activities etc. However, they are mostly longer term interest rates that matters in real economy as in case of mortgages for households or firms’ investments into fixed assets. These effects are often missing in recent models.

We can go further and think about economic policy authorities that influence real economy with their monetary and fiscal policy measures. First, the main objective of Central Banks is to maintain price stability and short term policy rate serves as a main tool. But economy is largely affected by long term interest rates, which are influenced not only by short term rates but also by term premia. For instance, interesting discussion of monetary policy rules, their impact on term premia and possible spillovers to real economy was brought by Kozicki & Tinsley (2002). Second, fiscal policy imposes rules and affects real economy by taxes and expenditures. Both policies also have an effect on term premia and different maturities along the yield curve as pointed out by Dai & Philippon (2005). For these reasons it is important to be aware of relationship between short and long term interest rates and their pass-through to the economy.

Macro-finance modelling that tries to connect these two spheres has been growing quite quickly. Attempts use mainly Vector-Autoregression (VAR) approach, which is purely based on statistics. In other words this approach uses estimated VAR to derive dynamics of the yield curve. Despite that these models are useful, they do not tell much about economic structure. Therefore use of structural model would be beneficial.

The paper attempts to employ DSGE model\(^1\) for several reasons. First, structural

\(^1\)Terms structural model and DSGE model are used interchangeably.
model gives more information about links in an economy, since they are based on economic theory. It does not have to rely on latent factors that are difficult to interpret directly. Second, DSGE models are widely used for macroeconomic analysis, so it is easier to extend these existing models that benefit from better information about structure of an economy than build separate macro-based model just for the yield curve analysis. Moreover, in the case of two models, it must be ensured that they would give consistent results and thus consistent opinion. As the paper shows, connecting both types of models with a financial part can be done in very similar way. Finally, having in mind that not only short term rates matters, this approach opens door for answering questions such as what impacts changes in the yield curve have on the real economy.

The aim of the paper is threefold. First, to show that it is possible to come up with a consistent derivation of financial model, including yield curve using Dynamic Stochastic General Equilibrium Approach applying basics of financial models. Consistency of macroeconomic and financial part is in derivation of pricing kernel equation, which is a central point for deriving yield term structure. It is done using macroeconomic variables and structural parameters only. It further allows to estimate the effect of basic macroeconomic variables such as the private and the government consumption, the short term interest rate and the inflation rate on the term structure of interest rates.

Second, using this approach, structural macro-finance model is able to fit real yield curve data. Such analysis of macroeconomic shocks to yield term structure should be of importance mainly for economic policy authorities. However, these institutions do not ignore the importance of analysis of the yield curve, it is usually done quite separately from the real economy analysis. Results of such a complex model would be consistent with their macro forecasts and could serve as a benchmark to other yield curve analysis.

In case of Central Bank, a simple analysis can give a notion to what extent will economic situation have an impact on different parts of yield curve and how long are these influences likely to persist and how costly possible interventions might be. The case of fiscal authority is for the purpose of illustration very simplified, however, it shows an essence of the issue, i.e. that for debt management it is crucial to understand the relationship between real economy and yield term structure, since it is important for maturity distribution of the debt.

Finally, the book is contributory also in a sense that it opens up important issues for future research. It would be very attractive to show impacts of term structure on the real economy as well, but the solution is not as straightforward as it may seem. To do this it would be necessary to establish and test proper links between agents in an economy and also deal with a problem of cyclical influences (from real economy to term structure and then back to economy).

The paper is organized as follows. Chapter 2 provides an overview of macro-finance
modelling, which serves as basics for this study.\footnote{An interested reader is referred to nice literature overview in Rudebusch \textit{et al.} (2007).} Chapter 3 introduces rather small scale DSGE model,\footnote{The reason for using smaller model is that it is much easier to show the derivation without loss of generality. Applying the approach to larger models stems from the same logic, however matrices and equations are uglier.} which is expanded with financial part. The underlying model includes four agents; households, firms, Central Bank and Government. Solution of the model results in a nonlinear form, which is then log-linearized in order to keep the system tractable. Further, the macro model is connected with financial part. The connection of these two models can be used for consistent derivation of the yield curve. Chapter 4 focuses on typically used VAR model, inspired mostly by Ang & Piazzesi (2003). Model with the same number of four macroeconomic variables is supplied with three latent factors influencing level, slope and curvature of yield curve. Chapter 5 brings an overview of the data used and compares the two models in terms of analyzing impacts of macroeconomic shocks to real economy and to yield curve. Chapter 6 continues with bringing results and shows implications of macroeconomic shocks on yield structure using two example cases: for monetary and fiscal policy. Chapter 7 summarizes and concludes.
2 Macro-finance models

There is quite large focus on macro-finance modelling in recent literature. Attempts to connect these two spheres have important implications for both type of models: finance models focused on asset pricing of securities and macroeconomic ones that serves mainly for economic policy analysis, simulations and forecasting. Generally, three groups of recent models can be distinguished. First, Affine VAR models, where a typical representatives are macro-finance models relying on econometric approach. Second group includes Affine DSGE based macro-finance models that try to connect structural models with financial part. Non-Affine DSGE based macro-finance models can be identified as a third group.

2.1 Introductory notes

Macro-finance models try to explain prices of various fixed income securities. Besides government bonds one can be interested in derivatives, such as swaps, futures on interest rates and others. A theory stems from basics of finance modelling. Bond prices can be expressed in terms of yields (denoted as $y^n_t$ for $n$-period bond)

$$y^n_t = -\ln \frac{Q^n_t}{n},$$

where $Q^n_t$ is price of $n$-period bond at time $t$. Obviously it is also possible to express forward rate ($f^n_t$) for such bonds:

$$f^n_t = \ln \frac{Q^n_t}{Q^{n+1}_t},$$

and respective yields

$$y^n_t = -\sum_{i=1}^{n-1} f^n_i.$$  

Short term rate then equals to $r_t = f^0_t = y^1_t$.

Asset pricing theory stems from basic equation saying that in arbitrage-free environment (a central point of macro-finance models) a positive number $M$ exists such as that for one period return $R_{t+1}$ following equation holds:

$$1 = \mathbb{E}_t[M_{t+1}R_{t+1}].$$

In other words, as mentioned in Cochrane (2005) “The asset pricing model says that, although expected returns can vary across time and assets, expected discounted returns

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4Excellent discussion of different types of models can be found in Rudebusch et al. (2007)

5The paper focuses on government bonds in order to explain an essence of the modelling approach rather than to deal with additional risks related with latter mentioned types of securities.
should always be the same, 1.” The number $M$ is denoted as pricing kernel and represents an important variable in these models since it allows to price any type of asset mentioned previously. As noted by Bernanke et al. (2004) assumption of no arbitrage in the bond market implies that a single pricing kernel determines the values of all fixed income securities.

When estimating bond prices, we can define one period return

$$R_{t+1} = \frac{Q^n_{t+1}}{Q^n_{t+1}},$$

and after substituting into the equation (4)

$$Q^n_{t+1} = \mathbb{E}_t[M_{t+1}Q^n_{t+1}].$$

Recursive approach allows to calculate prices of bonds with an initial condition $Q^0_t = 1$ saying that price of one dollar bond maturing today is worthy one dollar.

**Inspiration by Vasicek’s model**

Pioneering model, which is impossible to omit when talking about bond pricing is the one of Vasicek (1977) that occurred in two versions: simple one factor version and multiple factors that explain bond prices.

In his basic model, Vasicek uses one single factor\(^6\) – short term rate upon which prices of bonds depends. This variable (let’s denote it $x$) is assumed to follow AR(1) process.

$$x_{t+1} = \phi x_t + (1 - \phi)\bar{x} + \theta \epsilon_{t+1},$$

where $\bar{x}$ stands for mean of $x$, $\theta^2$ is variance and parameter $\phi$ determines the speed of mean reversion, in other words how quickly the process converges to its mean. Pricing kernel then satisfies condition that connects variation in explanatory variable with pricing factor determining future prices of bonds

$$\ln M_{t+1} = -x_t - \alpha + \Lambda \epsilon_{t+1},$$

where $\Lambda$ is called price of risk being the parameter determining covariance between shocks to explanatory variable $x$ and $M$. Parameter $\alpha$ equals to $0.5\Lambda$.\(^7\) This can be rewritten into final form:

$$\ln M_{t+1} = -x_t - 0.5\Lambda^2 + \Lambda \epsilon_{t+1}.$$  

\(^6\)We refer to the version in discrete time since it is used in the remainder of this paper. \(^7\)This stems from log-normality assumption of bond prices (and kernel). Generally for log-normal variable $Q$ with mean $\mu$ and variance $\sigma^2$ is true that $\ln \mathbb{E}_t[Q] = \mu + \frac{\sigma^2}{2}$. We know from equation 8 that $\mu = (-x_t - \alpha)$ and variance is $\Lambda^2$. Prices of bonds are solved recursively, so as mentioned, $Q^0_t = 1$ and $Q^1_t = \mathbb{E}_t[m_{t+1}] = R_{t+1}$. Then $\ln Q^1_t = -x_t + 0.5\Lambda^2 = -x_t$ (recall that $x_t$ is assumed to be short term rate) and thus $\alpha = 0.5\Lambda^2$ must hold.
Prices of bonds for different maturities are derived from general equation

\[- \ln Q^n_t = R^n_t = A_n + B_n x_t, \tag{10}\]

where intercept \((A_n)\) and slope \((B_n)\) are functions of parameters of the underlying model (from equation 7) and pricing kernel (equation 8).

### 2.2 Example models

Current models usually use multiple variables to describe yield developments. Already Knez et al. (1994) showed the importance of including more factors using an example of US money market. In his study, a three-factor model was able to explain about 86% and a four-factor one around 90% of variation in money market returns. Vasicek’s model also includes constant price of risk. This was extended by Cox et al. (1985) in famous CIR model that introduced time varying market price of risk. Current models use a non-constant price of risk when estimating pricing kernel. More specifically, market price of risk varies with underlying variable(s), i.e.

\[\Lambda_t = \Lambda_0 + \Lambda_1 x_t, \tag{11}\]

where \(x_t\) stands for explanatory variables. The crucial one is not a short term interest rate, but various macroeconomic variables can serve for this purpose. Also in this paper we stick to multiple factor version.

In fact, recent macro-finance models can also be viewed from two different perspectives. First, depending on the way how they derive prices of bonds. “Affine term structure models” assume bond prices to be a log-linear function of a vector of state variables, such as in (10), while “Non-affine term structure models” uses rather nonlinear function instead.

Second distinction classifies approaches depending on the type of underlying macro model. In other words, how they determine the explanatory variable \(x_t\) and its dynamics necessary for derivation of financial part of the model in (9)–(11). In this place, most approaches rely on VAR, while others tend to employ structural DSGE.

**Affine VAR based macro-finance models**

The first group of models uses AR process of explanatory variable for deriving the dynamics, which is very similar to Vasicek’s case. Since there is a difference in the number of variables used we are moving from simple AR process to VAR with inflation and production (mainly in terms of output gap) as usual variables. These models rely on econometric theory, so the choice of variables is often done by principal component

\(^8\)For the general case, please refer to Bolder (2001) for detailed discussion and derivation and Backus et al. (1998) for broader overview of various modifications of models.
method and number of lags is determined based on robustness of results, i.e. value added of these lags to VAR model dynamics.

One of the most cited models of this type is that of Ang & Piazzesi (2003), which includes 12 lags of variables from two groups: (i) inflation group represented by different price indices and (ii) activity group, with unemployment, vacancies and industrial production index. This simple model is extended by three so called unobservable (or latent) factors, which help to explain different characteristics of the yield curve, namely level, slope and curvature. They find that these basic macro factors are able to explain about 85% of variance at a short end and about 40% at a long-end of the yield curve.

In this group of models we may also mention Kozicki & Tinsley (2002) that focus more on different monetary policy rules and show that term premia is a function of policy rules parameters. Wu (2001) showed that monetary shocks derived from VAR model and/or from Taylor rule are in a strong correlation with latent factor determining slope of the yield curve.

**Affine DSGE based macro-finance models**

A good example of this group of models is the model of German economy presented in Hördahl et al. (2006) using quasi structural approach with inflation and output gap as explanatory variables. The drawback is mainly in implying restrictions on market price of risk to allow for some interactions between macro variables and prices of risk only. Price of risk is derived by an ad-hoc specification based on model fitting. Also Wu (2006) uses the structural approach and employs nominal and real rigidities built in a model. He consistently derives price of risk from households utility, which is the starting point of connecting macro model with financial one. Wu also confirms results of previously published unrestricted VAR models that monetary policy affects mainly the slope of the yield curve, while technology shock determines the level. However, the model includes some important attributes, derived term premia affecting yield curve dynamics is constant here, which is the main handicap.

**Non-Affine DSGE based macro-finance models**

Rudebusch & Swanson (2008) came up with a very challenging approach that ends up very close to the solution of time varying term premia. They use a medium scale DSGE model with nominal and real rigidities, which is then approximated by a nonlinear third order Taylor expansion. This approach is, as authors also admit, extremely demanding from the computational point of view even for relatively small model. For larger models, it would become hardly feasible.
3 Structural DSGE approach

This Section introduces general features of DSGE model that is then used for analysis\(^9\) and tries to explain the rationale behind. It shows the solution and derives the financial part with the term structure of interest rates the term premia.

3.1 DSGE model

Let us firstly discuss the substance of different blocks of the model with basic equations.

As a general equilibrium model, it consists of several blocks that are in mutual interactions. The main relationships between blocks and key variables can be seen in Figure 1.

![Figure 1: Overview of the model](image)

**Notation**

For better understanding the notation, variables marked by capital letter (e.g. \(X_t\)) are levels (billions of USD, thousands of people, etc.), lower case letters stands for natural logarithm of original variables (e.g. \(x_t \equiv \ln X_t\)) or some relative share (rate of unemployment, debt as a share of GDP, etc.).

3.1.1 Households

Households’ block focuses on analysis of consumers’ behaviour by splitting the budget of a household on consumption and savings. For this purpose, let’s have a representative

\(^9\)For more detailed explanation of some parts of the model, please refer to Appendix A.
consumer who faces an infinite horizon. In each period of time he aims at optimizing his consumption regarding the budget constraint given by his income.

The consumer is assumed to be a liquidity unconstrained, so he has unrestricted access to capital market. For the sake of simplicity, we have dropped a liquidity constrained households that follows a simple ‘Rule-of-thumb’ behaviour.\(^{10}\) In other words, the model relies on Ricardian equivalence, saying that consumers are highly connected to the future and they decide on the basis of permanent income theory\(^{11}\). Although the concept of Ricardian equivalence has been criticized, it should not be viewed as a drawback of the model. Some studies show that based on evidence, Ricardian equivalence cannot be rejected. E.g. Evans (1991) shows that however Ricardian equivalence does not have to be strictly valid in some rather extreme cases, it is still a good approximation. He tested this hypothesis also in Evans (1993) on the panel of 19 OECD countries and could not reject the hypothesis in case of 18 of those (United States included)\(^{12}\).

The optimization problem to split the consumers’ sources inter-temporally between consumption today and savings for tomorrow can be expressed by a utility function. There are many types of functions one can think of. It is important to have a unique solution, thus we accept the typical assumption of a concave utility function indicating declining marginal value of consumption. Moreover, a restriction on households’ preferences is used. In line with RBC models a balanced growth path is assumed (approximately constant consumption-growth ratio). The best solution for this class of models is to utilize the constant relative risk aversion utility function (CRRA), which was discussed and proved in detail by King et al. (1988).

Habit formation is incorporated according to Abel (1990) and Fuhrer (2000) and is defined as \(H_t^c = \gamma C_{t-1}\), where \(\gamma\) represents the habit persistence parameter, measuring the effect of the past consumption on current utility \((0 \leq \gamma \leq 1)\).\(^{13}\) The element has also an evidence in Hall (1979) who proves a significant contribution of one period lagged consumption to the analysis. Neither longer consumption lags nor other lagged variable (e.g. lagged income) added to the utility function improve the analysis.

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\(^{10}\)These agents do not have an access to a capital market and they consume all resources in each period of time. The concept is usually used to approximate the behaviour of certain groups of consumers – especially young people, low-skilled persons, retirees etc.

\(^{11}\)E.g. Ricardian consumers view taxes as an exact offset to the government debt, i.e. deficit financing (issuing debt) is equal to increase taxes and differs only in time when it is applied.

\(^{12}\)The case of the US is examined more in depth in Evans (1988).

\(^{13}\)Usually the persistence parameter (habit) is introduced also to the labour supply (or leisure). Since papers mentioned in this paragraph prove that the habit in consumption fits the data more accurately, Lettau & Uhlig (2000) conclude that introducing an additional habit into the labour does not affect the dynamics of the consumption so much (or even in slightly negative manner). Moreover, in such specification the labour input behaves counter-cyclically, which is not observed in the data. Labour habit also brings more difficulties into simultaneous explanation of financial markets and business cycle facts, which is the aim of the book.
There is also rich evidence from the data that proves the need for habit formation, e.g. in Ferson & Constantinides (1991). These authors study, whether the time non-separability of consumption should be attributed to a durability of consumption goods or the habit formation. They find more evidence for the latter. An interesting study is also Heien & Durham (1991) who use detailed households’ data to test the habit formation hypothesis. Although they attribute the habit effect predominantly to the aggregated consumption data rather than cross-section data, their overall conclusion is in favour of habit formation playing an important role in households’ behaviour.

Thus the optimization problem concentrates in following utility function:  
\[
\max_{\{C_t,N_t,B_t\}} U_\tau = \mathbb{E}_\tau \sum_{t=\tau}^{\infty} \beta^t \delta^c_t \left[ \frac{(C_t - H^c_t)^{1-\psi_c}}{1-\psi_c} - \frac{(N_t)^{1+\psi_n}}{1+\psi_n} \right],
\]  

(12)

where:
\( U_\tau \) lifetime utility function of a consumer,
\( \beta^t \) discount factor (captures impatience),
\( \delta^c_t \) preference shock,
\( C_t \) real consumption of household,
\( \mathbb{E}_\tau \) conditional expectations,
\( H^c_t \) habit level of consumption,
\( N_t \) labour supply,
\( \psi_c \) coefficient of risk aversion (reciprocal to el. of subst. of consumption),
\( \psi_n \) coefficient of risk aversion (reciprocal to el. of subst. of labour supply).

A consumer must respect the budget constraint which in our case has following form:
\[
Q_t B_t + (1 + \tau^p) P_tC_t = Q_{t-1} B_t + (1 - \tau^w + \tau^b) W_t N_t + (1 - \tau^f) \Pi_t,
\]  

(13)

where:
\( \tau^b \) rate of social benefits,
\( \tau^f \) corporate income tax rate,
\( \tau^p \) tax rate on production,
\( \tau^w \) personal income tax rate,
\( \Pi_t \) profit from firms,
\( B_t \) number of bonds,
\( P_t \) consumer price index,
\( Q_t \) price of bonds,
\( TR_t \) transfers to households,
\( W_t \) aggregate nominal wage.

\[14\] i.e. existence of some frictions in consumption.

\[15\] Utility function occurs in an additive form rather than multiplicative. The latter has an unfavourable implications on asset prices (derived in Section 3.3) through a negative correlation between consumption and leisure and thus low volatility of marginal rate of substitution, resulting in low term premia. For discussion, see Lettau & Uhlig (2000).
Solution is also consistent with No-Ponzi game condition.\footnote{It is worth noting that No-Ponzi game condition does not have direct impact on consumption dynamics, which is inferred from maximization of utility function (12), subject to two-period budget constraint (13). This dynamics represents short term cyclical deviations, while terminal condition, such as No-Ponzi game, is important for long term convergence of the model. Thus it imposes a restriction on steady state values. Having this in mind, No-Ponzi game condition is derived by rewriting the two-year period budget constraint (13) to infinity. It must apply that \( \lim_{t \to \infty} R_t B_t = 0 \), where \( R_t = [(1 + i_1)(1 + i_2)\ldots(1 + i_t)]^{-1} \) is a discount factor. In other words, impact of discount factor (steady state of interest rate \( \bar{i} \)) must be higher than that of income and consumption \( \bar{c} \) so the model converges and (14) is satisfied. Derivation of No-Ponzi game condition can be found in e.g. Niepelt (2011).}

Having specified the model one can derive the optimal behaviour of households. This relationship is not difficult to derive as a first order condition (FOC) from the utility function (12) and budget constraint (13). The problem concentrates in solving lagrangian by partial differentials which give us FOCs for both, consumption and labour supply.

The equation of consumption reveals determinants of relative consumption at present time \( t \) and future period \( t + 1 \) depending on the risk aversion factor \( \psi_c \). When considering the trade off, expected real interest rate and expected change in habit consumption\footnote{Which in fact signifies a change in the level of consumption from previous period \( t - 1 \) to current one \( t \) since we defined habit consumption as \( H^c_t = \gamma C_{t-1} \).} play role as follows

\[
1 = \beta^t \mathbb{E}_t \left[ \left( \frac{C_{t+1} - H^c_{t+1}}{C_t - H^c_t} \right)^{-\psi_c} \frac{1 + \bar{i}_t}{1 + \pi_t + \delta^c_{t+1}} \right] \tag{15}
\]

In the case of labour supply, households offer their work relatively to the level in previous period, with respect to the risk aversion factor \( \psi_n \). The result is influenced by real net income (relative to prices reflecting purchasing power) and the fact that households must respect the limited time for either work or leisure

\[
N^\psi_n_t = \frac{(1 - \tau^w + \tau^b) W_t}{(1 + \tau^p) P_t} (C_t - H^c_t)^{-\psi_c} \tag{16}
\]

\subsection*{3.1.2 Firms}

Introductory notes

Modelling of business sector is based on existence of competitive firms, that produce differentiated output \( Y_{it} \), aggregate it and sell to households and/or government.
In the following text, we assume a production of firms to be equal to the production of the whole economy $Y$. This seems to be relevant looking at the data concerning the US economy, which is illustrated by Figure 2. These time series are correlated with coefficient equal to 0.54. Moreover it should be borne in mind that the model works with deviations from steady states (logs) rather than levels.

Figure 2: GDP and industrial production

In this case, capital and investments are not specified directly. It is done in order to avoid non-negligible difficulties with estimating capital stock. However this does not necessary bias the model results. The great share of US GDP is in consumption (around 70%) rather than investment (17%). Moreover, from the point of dynamics, investments are highly correlated with consumption (mainly with durable goods), which illustrates Figure 3.

Correlation of these two time series is almost 70%. The only difference is in variation of data, with higher fluctuations of investments (standard deviation amounts to 11.6%) comparing to consumption (deviation of 2%). Therefore to keep the model structure simple and easily tractable, we follow the approach of modelling overall domestic demand (consumption and investment) altogether rather than separately.
Price setting

There is a continuum of firms operating in monopolistic market and producing a homogenous output\(^\text{18}\) \((Y_t)\) in the sense of using an identical technology. We employ standard assumption that an aggregate output \((Y_t)\) is defined by Dixit-Stiglitz constant elasticity of substitution aggregator (Dixit & Stiglitz (1977))

\[
Y_t \equiv \left[ \int_0^1 (Y_{it})^{\frac{\sigma - 1}{\sigma}} \, di \right]^{\frac{\sigma}{\sigma - 1}},
\]

where:
- \(\sigma\) elasticity of substitution among goods produced,
- \(Y_{it}\) individual good in production of a firm,
- \(Y_t\) demand for the total production.

The same assumption applies to the price level \((P_t)\), i.e.

\[
P_t \equiv \left[ \int_0^1 (P_{it})^{1-\sigma} \, di \right]^{\frac{1}{1-\sigma}}.
\]

\(^{18}\)It does not necessarily mean a production of a single good but rather a similar bunch of heterogeneous goods. It is no intention to study terms of trade between individual companies, so it is perfectly sufficient to introduce one representative firm. It would be possible, without any doubt, to have an elaborated sector with various types of firms with different production technologies. As an example can serve model GIMF of the IMF (Kumhof \textit{et al.} (2010)). However, this would have only a limited value added to our analysis.
Cost minimization in production of a unit of output implies the demand for each respective good

$$Y_{it} = Y_t \left( \frac{P_{it}}{P_t} \right)^{-\sigma},$$  \hspace{1cm} (19)

where:
- $P_{it}$ price of individual good,
- $P_t$ aggregate price level.

Firms produce their output using Cobb-Douglass production function

$$Y_{it} = \delta_i^z (N_t)^\alpha,$$  \hspace{1cm} (20)

with a single input factor of labour supply $N_t$. $\delta_i^z$ is a technology shock and $\alpha$ is a parameter.

The output is sold either to households or government, i.e.

$$Y_t = C_t + G_t,$$  \hspace{1cm} (21)

where:
- $C_t$ household consumption,
- $G_t$ government consumption.

Price setting of firms is derived based on rationale of the new Phillips curve employing real marginal costs instead of output gap in the place of the real activity measure. According to Calvo (1983), firms are allowed to reset their prices only when they receive some random signal. The probability of signal occurrence in each period of time is $(1 - \xi)$. Thus the parameter $\xi$ represents the frequency of price adjustment or flexibility.\(^\text{19}\).

Firms are thus separated on those that keep their price from the previous period $P_{it} = P_{it-1}$ and those who re-optimize the price. Aggregate price index, which is also an aggregate of prices of individual products, evolves according to

$$P_t = \left[ \int_0^1 (P_{it})^{1-\sigma} \, di \right]^{\frac{1}{1-\sigma}},$$  \hspace{1cm} (22)

where

$$P_{it} = \left[ (1 - \xi)(P_{it}^*)^{1-\sigma} + \xi(P_{it-1})^{1-\sigma} \right]^{\frac{1}{1-\sigma}}.$$  \hspace{1cm} (23)

Firms that update prices of their production do so following either backward looking or forward looking rule. In this approach we stem mainly from arguments of Gali

\(^{19}\)Prices are perfectly flexible when $\xi = 0$. In this case firms set their price in proportional relation to the movements in marginal costs.
Gertler (1999) who tested various approaches to estimate New Keynesian Phillips Curve. The reoptimizing price index is thus decomposed into two others

\[ P_{it}^* = \chi P_{it}^b + (1 - \chi)P_{it}^f \]  

(24)

A fraction of firms (\( \chi \)) reset prices using simple backward looking rule and update lagged optimal price of their competitors using inflation such as

\[ P_{it}^b = (1 + \pi_{it-1})P_{it-1}^*. \]  

(25)

It is worth noting that backward looking firms look purely to the past and use past inflation rate (\( \pi_{t-1} \)) to update optimized prices from previous period (\( P_{it-1}^* \)).

The rest of firms (\( 1 - \chi \)) are forward looking oriented and set their prices regarding discounted sum of expected future incomes minus costs of production. In other words, each producers maximize profit function

\[ \max_{\{P_{it}, N_t\}} \Pi_{it}^P = \mathbb{E}_t \sum_{t=\tau}^{\infty} (\beta \xi)^t (P_{it} - MC_t)Y_{it} \]  

(26)

with respect to production technology (20) and demand (19).

The resulting FOC is following

\[ P_{it}^f = \frac{-\sigma}{1 - \sigma} \left[ \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\beta \xi)^j MC_{t+j}Y_{it+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\beta \xi)^j Y_{it+j}} \right], \]  

(27)

where \(-\sigma/(1 - \sigma)\) is a mark-up to marginal costs. The latter is assumed to be identical for every firm, since a perfect competition on the side of inputs is assumed. The only costs for firms are those of labour, so marginal costs (\( MC \)) are derived as unit labour costs from production function (20) such as

\[ MC_t = \frac{W_t}{\frac{\delta Y_t}{\delta N_t}} = \frac{1}{\alpha} \frac{W_tN_t}{Y_t}, \]  

(28)

with wages cleaning the labour market.\(^21\)

\(^{20}\)They show that the original output gap in the Phillips curve has an unwanted practical implications (beside the fact that output gap is unobservable) when estimating the curve, such as that the inflation leads the output gap. Not even a hybrid modification including a lagged inflation term brought substantial improvement. It is true mainly in quarterly models where the estimates shows insignificant results of the effect of output gap on inflation.

\(^{21}\)For the sake of simplicity we do not explicitly specify the labour market, i.e. deviations from steady states of labour demand and labour supply equals. In the model that works with deviations from steady state, levels of unemployment do not really matter. Loosely speaking, the unemployment does not enter to the model, but it is replaced by a preference of leisure in utility function (12). In following part, it allows to substitute for the wage using the optimum labour supply from households' decision and optimum labour supply regarding the firms’ production technology. The wage represent an average of the whole economy.
It should be mentioned that there is only a slight difference between backward and forward looking firms. When the inflation is stationary it converges to optimal behaviour. However there is a strong evidence from Gali & Gertler (1999) that incorporating backward looking behaviour improves the analysis by allowing for additional price stickiness.

3.1.3 Fiscal policy

The purpose of the Fiscal policy block is to derive the link between fiscal policy and real economy, and at the same time to set simple FP rule. First, it is useful to define revenues, expenditures and deficit and debt formation.

Revenues, expenditures and debt

Revenues of the government budget consist of several categories, mainly of taxes and other revenues.

\[ GR_t = PIT_t + CIT_t + PROD_t = \tau^w W_t N_t + \tau^f \Pi_t + \tau^p P_t Y_t + \delta^R, \]  

(29)

A significant part of taxes are those from income. Government revenues \((GR)\) from a personal income tax \((PIT)\) depend on implicit personal income tax rate \(\tau^w\) and on wages and employment \((W_t N_t)\). A corporate income tax \((CIT)\) is determined by an implicit rate \(\tau^f\) and operational surplus that firms get from their production after adjustment for wages (so \(\Pi_t = P_t Y_t - W_t N_t\)). Important part of revenues are also taxes imposed on production \((PROD)\), which are assumed to be a product of nominal production \((P_t Y_t)\) and implicit tax rate \(\tau^p\). \(\delta^R\) is a shock to government revenues.

Implicit tax rates are defined as a share of government income from each tax on its respective base, which is illustrated by following equations

\[ \tau^w = \frac{T^w_t}{W_t N_t}, \quad \tau^f = \frac{T^f_t}{\Pi_t}, \quad \tau^p = \frac{T^p_t}{P_t Q_t}. \]  

(31)

where \(T^w_t, T^f_t\) and \(T^p_t\) are total incomes from taxes on wages, corporate income and production respectively. Due to a stable development of these ratios in time (see Figure 4), the implicit rates are kept constant in the model. Even though implicit tax rate on wages seems little bit more volatile, the illustration shows that this is also the case of implicit rate of benefits but in inverse direction. Since the wage tax contributes to revenue side, benefit rate to expenditure side and both are calculated using the same basis, their fluctuation cancels out during the further derivation of the model.\(^{22}\)

Government primary expenditures contain government consumption \((G^c_t)\) and current transfer payments \((G^t_t)\). The latter is determined by implicit rate of benefits \(\tau^b\) and

\(^{22}\)Correlation coefficient of their respective growth rates is equal to -0.5. Regarding rather low importance of differences between these two parameters in the model, we can assume this to be sufficiently high negative correlation.
wage development. And $\delta_t^e$ stands for a shock to government expenditures.

$$GE_t = G_t P_t + \tau^b W_t N_t + \delta_t^e$$  \hspace{1cm} (32)

Not all budgetary items could be assigned to the mentioned categories. Therefore some of them that are of minor importance and difficult to consistently implement into the model are excluded. Revenue side is thus covered by 70% and expenditure side by 94%.

Having derived main equations for revenues and expenditures, it is easy to state the equation for the debt, which is a result of government’s balance of current year ($GE_t - GR_t$), level of government debt reached in previous period ($B_{t-1}$) and interest payments for such outstanding debt ($i_t B_{t-1}$).

$$B_t = GE_t - GR_t + (1 + i_{t-1}) B_{t-1}.$$  \hspace{1cm} (33)

**Fiscal rule**

Differences between expenditures and revenues and also current level of government debt may have an adverse impact on debt dynamics and thus on fiscal solvency. Fiscal rule should prevent this unfavourable development by ensuring the debt ratio not to explode.

Two important questions must be answered when introducing the fiscal rule into the model. First, what will be the reference variable that will activate the fiscal rule. Usually this role plays either debt or deficit, which are of main importance when assessing
the fiscal solvency. It seems that both targets could be mutually consistent\textsuperscript{23} when parameters of fiscal reaction function are adjusted. Both rules give comparable results as shown e.g. in Mitchell \textit{et al.} (2000).

Second issue concerns a budgetary item that should be adjusted by the fiscal rule. Unfortunately, there is no clear evidence in economic literature which item should play the role. Generally, most analysis rely on tax rules where fiscal policy rectifies the debt dynamics by changes in tax rates. Unluckily various difficulties are related with introducing tax rules into the model (an optimal taxation problem, omitted interactions with monetary policy and internal consistency of the model).

Probably the best way how to deal with difficulties is to introduce an expenditure fiscal rule, which is much easier, more flexible (regarding legislative process) and does not have an impact on relative prices.

So the budgetary item, adjusted by the fiscal rule here is represented by government expenditures on consumption \((G_t)\) that is a weighted average of lagged value and “optimal” consumption

\[ G_t = (1 - \phi_g)G^o_t + \phi_g \frac{G_{t-1}}{P_{t-1}}. \]  

(34)

where:

- \( G_t \) government consumption,
- \( G^o_t \) “optimal” consumption,
- \( \phi_g \) degree of sluggishness.

The “optimal” government consumption adjusted by the fiscal rule is derived from an assumption of balanced primary government budget (zero primary deficit)\textsuperscript{24} in equilibrium, i.e. \( D_t = GE_t - GR_t = 0 \). So the result is following

\[ G^o_t = (\tau^w - \tau^b) \frac{W_t N_t}{P_t} + \tau^Y \frac{P_Y - W_t N_t}{P_t} + \tau^p Y_t + \delta^g_t. \]  

(35)

where \((\delta^g_t = \delta^R_t + \delta^E_t)\) is government balance shock. The first element is zero, since implicit rates of wage tax and benefits are equal as could be inferred from the Figure 4 and related discussion.

3.1.4 Monetary policy

Monetary policy is conducted by the Central Bank that maintains price stability in the economy mainly through affecting short term interest rates. In doing so, the Bank

\textsuperscript{23}However the deficit and debt oriented rules are consistent, there might be some difficulties to take the model to the data since sum of deficits does not always equal to the change of debt. For discussion see e.g. Dvořák (2010).

\textsuperscript{24}Primary deficit is used in order to avoid possible fiscal restriction that would occur (i) in case of monetary expansion through interest rate payments or (ii) in case of higher inflation pressures.
analysis macroeconomic indicators, such as output gap and inflation. This can be formalized using modified Taylor rule, developed by Taylor (1993) and extended by Svensson (2000), which is standard specification in structural models.

\[ i_t = (1 - \phi_i)(\bar{i} + \lambda_{\pi}\hat{\pi}_t + \lambda_y\hat{y}_t) + \phi_i i_{t-1} + \delta_t, \]  

(36)

where:
- \( \bar{i}_t \) steady state short term nominal interest rate,
- \( i_t \) short term nominal interest rate,
- \( \hat{\pi}_t \) deviation of the inflation rate from its target value,
- \( \hat{y}_t \) output gap,
- \( \lambda_{\pi}, \lambda_y \) policy parameters,
- \( \delta_t \) monetary policy shock,
- \( \phi_i \) interest rate smoothing parameter.

The main advantage of the extended Taylor rule is the interest rate smoothing, which improves the performance and robustness of the original rule for many reasons. First, the smoothing reduces different kinds of uncertainty stemming from data, model specifications and/or its parameters. Secondly, it reflects gradual reaction of Central Banks in adjusting the interest rates as a reaction to persistency of shocks into the economy. And thirdly, following the previous it ensures a stability of financial market and formation of expectations in the economy.\(^{25}\) These reasons are supported also by Levin et al. (1999), who tested robustness of simple monetary rules. After testing different rules on the US economy, they conclude that more complicated rules does not have a substantial value added in stabilizing the inflation and output comparing to simple rules. They also suggest relatively high degree of smoothing with \( \phi_i \to 1 \). However, it seems to be the case of large and closed economies, since Côté et al. (2004) provides different results for Canadian economy.

### 3.2 Solution

#### 3.2.1 Log-linearization

Resulting FOCs equations often take nonlinear forms, which is not easy to deal with and solution may be very sensitive to small deflects in variables. Therefore it is useful express equations in the form of deviations from steady states using log-linearization.\(^{26}\)

Generally, the approach can be formalized by an example of some nonlinear function with \( X_t \) representing endogenous and \( Z_t \) exogenous variable

\[ X_t = F(X_{t-1}, Z_t), \]  

(37)

rewritten into a logarithmic form using

\[ e^{x_t} = F(e^{x_{t-1}}, e^{z_t}), \quad x_t = \ln F(x_{t-1}, z_t) \]  

(38)

\(^{25}\)Further discussion can be found in Srour (2001).

\(^{26}\)E.g. Fuhrer (2000) concludes his analysis with comment, that solving nonlinear model and its linearized version lead to “nearly identical results”.

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and finally
\[ x_t = f(x_{t-1}, z_t), \quad (39) \]
where lower case letters are logarithms of respective variables. Then, the function can be easily approximated by the first order Taylor expansion as follows
\[ x_t \approx f(\bar{x}, \bar{z}) + \left( \frac{\partial f(\bar{x}, \bar{z})}{\partial x_{t-1}} \right) (x_{t-1} - \bar{x}) + \left( \frac{\partial f(\bar{x}, \bar{z})}{\partial z_t} \right) (z_t - \bar{z}), \quad (40) \]
where \( \bar{x} \) and \( \bar{z} \) are steady states of variables \( \partial f(\cdot)/\partial x_t \) and \( \partial f(\cdot)/\partial z_t \) denote elasticities.

**Steady state values**

Steady state values represent long term equilibrium of the model. The result is a trend component of each variable; constant steady state for stationary variables (interest rate, unemployment rate, etc.) and fixed growth rate for non-stationary variables (GDP, consumption, etc.).

\[ 1 = \beta(1 + \bar{i} - \bar{\pi}), \quad (41a) \]
\[ \bar{N}^{\psi_c} = \frac{W}{P} \left[ (1 - \gamma)\bar{C} \right]^{-\psi_c} \quad (41b) \]
\[ \bar{Y} = \bar{N}^\alpha, \quad (41c) \]
\[ \bar{P}^f = -\frac{\sigma}{1 - \sigma} \bar{MC} \quad (41d) \]
\[ \bar{MC} = \alpha \frac{W \bar{N}}{\bar{Y}} \quad (41e) \]
\[ \bar{Y} = \bar{C} + \bar{G}. \quad (41f) \]

**Log-linearized model**

Derived equations take the form of deviations from steady states. The optimal households’ consumption takes form
\[ \dot{c}_t = \frac{1}{1 + \gamma} \dot{c}_{t+1} - \frac{\gamma}{1 + \gamma} \dot{c}_{t-1} - \frac{1 - \gamma}{\psi_c(1 + \gamma)} \left[ \dot{e}_t - E_t(\bar{\pi}_{t+1}) \right] \]
\[ + \frac{(1 - \gamma)(1 - \rho_c)}{\psi_c(1 + \gamma)} \dot{\delta}_t, \quad (42) \]
where habit parameter \( \gamma \) is crucial parameter for IS curve dynamics. Interest rates elasticity of consumption then depends on both, habit parameter \( \gamma \) and risk aversion
ψc. ˆδt stands for preference shock. Households’ decision about optimal labour supply is as follows

\[ \hat{n} = \frac{1}{\psi_n} (\hat{w}_t - \hat{p}_t) + \frac{\psi_c}{\psi_n(1 - \gamma)} (\gamma \hat{c}_{t-1} - \hat{c}_t). \]  

(43)

The Phillips curve takes standard form, dependent on lagged and lead inflation and real marginal costs

\[ \hat{\pi}_t = \frac{\chi}{\xi + \chi[1 - \xi(1 - \beta)]} \hat{\pi}_{t-1} + \frac{\beta \xi}{\xi + \chi[1 - \xi(1 - \beta)]} \hat{\pi}_{t+1} 
+ \frac{(1 - \chi)(1 - \xi)(1 - \beta \xi)}{\xi + \chi[1 - \xi(1 - \beta)]} r\hat{mc}_t, \]  

(44)

where \( \xi \) and \( \chi \) determine dynamics of the Phillips curve. Real marginal costs are specified as follows

\[ r\hat{mc}_t = \frac{\psi_c}{(1 - \gamma)} (\hat{c}_t - \gamma \hat{c}_{t-1}) + \left( \frac{1 + \psi_n}{\alpha} - 1 \right) \hat{y}_t - \frac{1 + \psi_n}{\alpha} \hat{\delta}_t. \]  

(45)

the last element \( \hat{\delta}_t \) stands for technology shock.

The model is closed by functional specification of authorities. For government consumption we end up with equation

\[ \hat{g}_t = (1 - \phi_g)[\mu_g(1 - \sigma) r\hat{mc}_t + \hat{y}_t + \hat{\delta}_t] + \phi_g \hat{g}_{t-1}. \]  

(46)

And for monetary policy we use specification of Taylor rule including interest rate smoothing

\[ \hat{i}_t = (1 - \phi_i)(\lambda_\pi \hat{\pi}_t + \lambda_y \hat{y}_t) + \phi_i \hat{i}_{t-1} + \hat{\delta}_i. \]  

(47)

Total production of the economy equals to weighted sum of private and government consumption, expressed in deviations

\[ \hat{y}_t = \omega_{yc} \hat{c}_t + \omega_{yg} \hat{g}_t, \]  

(48)

where \( \omega_{yc} \) (resp. \( \omega_{yg} \)) is a share of private (resp. government) consumption on GDP.

### 3.2.2 Solution of the model

Solution of the model lies in solving the system of linearized equations. Since parameters by the respective variables are sometimes complicated, we simplify the equations by substitution for parameters with omegas\textsuperscript{27} (\( \omega \)). Resulting system of equations in the

\textsuperscript{27}A full list of substituted parameters can be found in Appendix B.
form of deviations from the steady states is as follows

\[
\hat{c}_t = \omega_{cf}\hat{c}_{t+1} - \omega_{ct}\hat{c}_{t-1} - \omega_{ci}(\hat{i}_t - \hat{\pi}_{t+1}) + \omega_{cz}\hat{\delta}_t^c, \tag{49a}
\]

\[
\hat{g}_t = \omega_{gc}\hat{c}_t - \omega_{gct}\hat{c}_{t-1} + \omega_{gg}\hat{g}_{t-1} + \omega_{gzz}\hat{\delta}_t^g - \omega_{gzz}\hat{\delta}_t^z, \tag{49b}
\]

\[
\hat{i}_t = \omega_{ip}\hat{\pi}_t + \omega_{ic}\hat{c}_t + \omega_{ig}\hat{g}_t + \omega_{it}\hat{\pi}_{t-1} + \hat{\delta}_t^i, \tag{49c}
\]

\[
\hat{\pi}_t = \omega_{ppl}\hat{\pi}_{t-1} + \omega_{ppf}\hat{\pi}_{t+1} + \omega_{pc}\hat{c}_t - \omega_{pcl}\hat{\pi}_{t-1} + \omega_{pg}\hat{g}_t - \omega_{pze}\hat{\delta}_t^z, \tag{49d}
\]

\[
\hat{\delta}_t^c = \rho_c\hat{\delta}_{t-1}^c + u_t^c, \tag{49e}
\]

\[
\hat{\delta}_t^g = \rho_g\hat{\delta}_{t-1}^g + u_t^g, \tag{49f}
\]

\[
\hat{\delta}_t^i = \rho_i\hat{\delta}_{t-1}^i + u_t^i, \tag{49g}
\]

\[
\hat{\delta}_t^z = \rho_z\hat{\delta}_{t-1}^z + u_t^z. \tag{49h}
\]

There are four endogenous and four exogenous variables\(^{28}\) in the model. As apparent, equations consist of both, lagged (backward looking) and lead (forward looking) variables. One way to solve the system stems from the paper of Uhlig (1998) using the method of undetermined coefficients. First step is to rewrite equations into a matrix form

\[
G\dot{x}_t = F\dot{\pi}_t(x_{t+1}) + H\dot{\pi}_{t-1} + L\delta_t, \tag{50a}
\]

\[
\dot{\delta}_t = N\hat{\delta}_{t-1} + u_t; \quad \dot{\pi}_t = [u_t^c, u_t^g, u_t^i, u_t^z], \tag{50b}
\]

where \(\dot{x}_t = (\hat{c}_t, \hat{g}_t, \hat{i}_t, \hat{\pi}_t)^T\) is \((4 \times 1)\) vector of endogenous variables expressed in deviations from steady state, \(\hat{\delta}_t = (\hat{\delta}_t^c, \hat{\delta}_t^g, \hat{\delta}_t^i, \hat{\delta}_t^z)^T\) is vector of exogenous variables expressed in deviations from steady state and \(u_t = (u_t^c, u_t^g, u_t^i, u_t^z)^T\) is vector of innovations. \(G, F, H, L, N\) are matrices of structural parameters such as

\[
G = \begin{bmatrix}
1 & 0 & \omega_{ci} & 0 \\
-\omega_{gc} & 1 & 0 & 0 \\
-\omega_{ic} & -\omega_{ig} & 1 & -\omega_{ip} \\
-\omega_{pc} & -\omega_{pg} & 0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
\omega_{cf} & 0 & 0 & \omega_{ci} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
-\omega_{cl} & 0 & 0 & 0 \\
-\omega_{gcl} & \omega_{ggl} & 0 & 0 \\
0 & 0 & \omega_{ii} & 0 \\
-\omega_{pcl} & 0 & 0 & \omega_{ppl} \\
\end{bmatrix}, \quad \begin{bmatrix}
\omega_{cz} & 0 & 0 & 0 \\
0 & \omega_{gzz} & 0 & -\omega_{gzz} \\
0 & 0 & \omega_{iz} & 0 \\
0 & 0 & 0 & -\omega_{pz} \\
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
\rho_c & 0 & 0 & 0 \\
0 & \rho_g & 0 & 0 \\
0 & 0 & \rho_i & 0 \\
0 & 0 & 0 & \rho_z \\
\end{bmatrix}
\]

\(^{28}\)These are shocks to the model, which are assumed to follow a simple AR(1) process.
Having the model in form of matrices, we derive explicit solution based on solving a matrix quadratic equation. The solution\(^{29}\) is based on the recursive law of motion which ensures a stable solution

\[
\hat{x}_t = P\hat{x}_{t-1} + Q\hat{\delta}_t, \quad (51a)
\]

\[
\hat{\delta}_t = N\hat{\delta}_{t-1} + u_t, \quad (51b)
\]

where \(P, Q, N\) are matrices of reduced form parameters. These equations are important, since they allows to get rid of forward looking variables in (50).\(^{30}\) Finally we can express them in the concise form of VAR(1)

\[
\hat{y}_t = \Phi_1\hat{y}_{t-1} + \Theta_1\epsilon_t, \quad (53)
\]

where \(\hat{y}_t = (\hat{x}'_t, \hat{\delta}'_t)'\) and \(\epsilon_t = (0'_t, u'_t)'\) is a vector of errors. Matrices \(\Phi_1\) and \(\Theta_1\) consists of deep parameters of the model.\(^{31}\)

### 3.3 Affine term structure macro-finance model

In deriving financial part of the model I stem from rationale of asset pricing models. We focus on analysis of government bonds,\(^{32}\) which price \(Q^n_t\) for \(n\)-period bond is given by

\[
Q^n_t = \mathbb{E}_t(M_{t+1}Q^{n-1}_{t+1}) = \mathbb{E}_t\left[\prod_{j=1}^{n} M_{t+j}\right], \quad (54)
\]

where \(M_t\) is a stochastic discount factor (time varying pricing kernel or marginal rate of substitution) representing the difference in price of bonds between two periods of time,

\[^{29}\text{We do not focus explicitly on these transformation in detail since it is not the aim of the paper. However, an interested reader can find detailed description in Uhlig (1998) or more general discussion about solutions of dynamic linearized systems in Hansen (1985).}\]

\[^{30}\text{To be more specific, we stem from equations (50) and substitute for }(\hat{x}_{t+1})\text{ with (51a) and then for }(\hat{\delta}_{t+1})\text{ with (51b). We arrive to the solution of reduced form parameters matrices through}

\[
\hat{x}_t = -(G + FP)^{-1}H\hat{x}_{t-1} - (G + FP)^{-1}(FQN + L)\hat{\delta}_t,
\]

from which we can easily read the solution for \(P_{(4x4)}, Q_{(4x4)}\) matrices

\[
P = -(G + FP)^{-1}H,
Q = -(G + FP)^{-1}(FQN + L).
\]

\[^{31}\text{Derived matrices take following form}

\[
\Phi_1(8x8) = \begin{bmatrix} P & QN \\ 0 & N \end{bmatrix}, \Theta_1(8x8) = \begin{bmatrix} 0 & Q \\ 0 & I \end{bmatrix}.
\]

\[^{32}\text{T.-Constant maturity.}\]
i.e.
\[ M_{t+1} = \frac{Q^n_t}{Q^n_{t+1}} = \beta^t \mathbb{E}_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \right], \quad (55) \]
where \( \lambda \) is a parameter of lagrange function.

In order to analyze the term structure of interest rates we need to derive two main components: (i) pricing kernel \( (M_{t+1}) \), (ii) price of bonds \( (Q^n_t) \) and (iii) respective yields \( (r^n_t) \).

### 3.3.1 Pricing kernel

#### Specification in financial models

As mentioned in the Chapter 2, finance literature specifies the pricing kernel by
\[ M_{t+1} = \exp(-i_t - 0.5 \Lambda'_t \Lambda_t - \Lambda'_t \epsilon_{t+1} + \eta_{t+1}) \quad (56) \]
for multifactor affine models, e.g., in Dai & Singleton (2002). \( \epsilon_{t+1} \) stands for shocks into the economy, \( \eta_{t+1} \) is specific shock to pricing kernel and finally \( \Lambda_t \) is market price of risk connected with uncertainty about shocks into the economy
\[ \Lambda_t = \Lambda_0 + (\Lambda_1 \tilde{y}_t)\alpha, \quad (57) \]
where \( \alpha \in \{0, 0.5, 1\} \) in order to get affine (linear) term structure of interest rates and \( \tilde{y}_t \) is a vector of macroeconomic variables (from equation 53). The former restriction is made in order to have the affine form of term structure of interest rates. This is a general model that can be easily transformed into other well known models. By setting \( \Lambda_1 = 0 \), the model equals to multifactor Vasicek model (Vasicek (1977)). In case \( \alpha = 0.5 \) and \( \Lambda_1 \neq 0 \) the model becomes multifactor CIR model proposed in (Duffie & Kan (1996)).

#### Macroeconomic specification of kernel

One of the main objectives is to derive a link between “financial specification” of the stochastic discount factor (56) and discount factor from the macroeconomic model (55). Pricing kernel in macro-finance model can be inferred from households optimization problem.33

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33The derivation is quite straightforward from (12) and (13) from where we have
\[ \frac{\delta L}{\delta B_t} = 0 - \lambda_t \beta' Q_t + \mathbb{E}_t[\lambda_{t+1} \beta^{t+1} Q_{t+1}] . \]
Note that this problem was solved also for households’ decision about consumption today and tomorrow, which resulted in FOC (15). In that case when households decide about consumption or investments into 1-period bonds, it applies that \( M_{t+1} = \frac{1}{1+i_t} \).
In the equation (55) we can substitute for $\lambda_{t+1}$ and $\lambda_t$ from consumption optimization function,\(^{34}\) which results in the definition of pricing kernel, a function of marginal rate of substitution

$$M_{t+1} = \beta \mathbb{E}_t \left[ \left( \frac{C_{t+1} - H_{c_{t+1}}^c}{C_t - H_{c_t}^c} \right)^{-\psi_c} \frac{1}{1 + \pi_{t+1}} \frac{\delta_{t+1}^c}{\delta_t^c} \right]. \quad (58)$$

For detailed description see e.g. Cochrane (2005).\(^{35}\)

We can also write the equation in logarithms denoted by lower case letters.

$$M_{t+1} = \exp(-r_t + \pi_{t+1} + (\delta_{t+1}^c - \delta_t^c) - \psi_c \ln(c_{t+1} - \gamma c_t) + \psi_c \ln(c_t - \gamma c_{t-1})), \quad (59)$$

where $\ln \beta \equiv -r$ is real interest rate. We obtain a definition of pricing kernel similar to the financial one in (56). It is clear that discount factor depends on lagged, current and forward looking variables. We can gather them in a single vector $k_t$ of macroeconomic explanatory variables so that

$$k_t = (c_{t+1}, \pi_{t+1}, \delta_{t+1}^c, c_t, \delta_t^c, c_{t-1})', \quad (60)$$

The pricing kernel, needed to model the price of bonds, described by the equation (59) takes a nonlinear form, which would be quite difficult to solve. On the other hand it is not possible to use simple log-linear approach since it would lead to constant term premia. Therefore we will approximate the function by second order Taylor expansion around non-stochastic steady state in order to get time varying term premia.\(^{36}\)

$$M_{t+1} \approx \bar{M}_{t+1} + \bar{g}'(k_t - \bar{k}_t) + \frac{1}{2}(k_t - \bar{k}_t)'\bar{H}(k_t - \bar{k}_t), \quad (61)$$

\(^{34}\)Optimal consumption can be obtained by

$$\frac{\delta L}{\delta C_t} = \beta^t \delta_t^c (C_t - H_t^c)^{-\psi_c} - \lambda_t (1 + \tau^p) P_t.$$

From this a reduced form equation for $\lambda_t$ is easy to get as well as $\lambda_{t+1}$ by shifting variables for 1 period ahead.

\(^{35}\)As mentioned there at the beginning of Chapter 1: “The marginal utility loss of consuming a little less today and investing the result should equal the marginal utility gain of selling the investment at some point in the future and eating the proceeds. If the price does not satisfy this relation, the investor should buy more of the asset.” These consumption based models thus arrive to the pricing equation which is (54), where

$$M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}.$$

This, in our model, equals to (58).

\(^{36}\)It is important to derive time varying term premia. This may be ensured by using third order Taylor expansion for pricing kernel, or by assuming log-normal distribution of bond prices (i.e. normal distribution of macro variables) and second-order Taylor expansion of pricing kernel. The former leads to non-affine term structure and thus to computational difficulties as in e.g. Rudebusch & Swanson (2008).
where \( g \) is gradient (vector of partial derivatives) and \( H \) is Hessian (matrix of second order derivatives). This, when expressed in the form of deviations from steady state, takes the form

\[
\hat{m}_{t+1} = g'\hat{k}_t + \frac{1}{2} \hat{k}_t' H \hat{k}_t, \tag{62}
\]

Since the vector \( \hat{k}_t \) consists of macroeconomic variables, it is possible to substitute from the solution of macro model (53) using respective variables from \( \hat{y}_t \) applying choosing vector and get macroeconomic specification of pricing kernel.

**Useful transformation of macroeconomic model**

Because the solution of pricing kernel includes two different lags for some parameters, it is useful to extend the solution of macro model by one lag (in order to reduce the computational form) as follows:

\[
\hat{y}_t = \Phi_1 \hat{y}_{t-1} + \Theta_1 \epsilon_t, \tag{63a}
\]

\[
\hat{y}_{t-1} = I \hat{y}_{t-1}. \tag{63b}
\]

Or in matrix form

\[
\begin{bmatrix}
\hat{y}_t \\
\hat{y}_{t-1}
\end{bmatrix} = \begin{bmatrix}
\Phi_1 & 0 \\
I & 0
\end{bmatrix} \begin{bmatrix}
\hat{y}_{t-1} \\
\hat{y}_{t-2}
\end{bmatrix} \begin{bmatrix}
\Theta_1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\epsilon_t \\
0
\end{bmatrix}, \tag{64}
\]

which can be simplified as

\[
\hat{w}_t = \Phi \hat{w}_{t-1} + \Theta \varphi_t. \tag{65}
\]

**Consistent solution of kernel**

Combining the macro solution (65) with specification of pricing kernel described by (56) and (57), financial equation of kernel can be rewritten as

\[
\hat{m}_{t+1} = -(\Lambda_0' A_1 + e_3') \hat{w}_t - \Lambda_0' \varphi_{t+1} - \frac{1}{2} \hat{w}_t Q_1' A_1 \hat{w}_t - \frac{1}{2} \Lambda_0' A_0 - \hat{w}_t' A_1' \varphi_{t+1}, \tag{66}
\]

where \( e_3' \) is a choosing vector\(^{37}\) denoting the interest rate in vector of macro variables \( \hat{w} \).

At the same time I have derived macroeconomic solution of kernel in (62) using variables and parameters of macroeconomic model. After substituting to this equation for \( g, k_t \) and \( H \), I can write final solution as follows\(^{38}\)

\[
\hat{m}_{t+1} = \Omega_w \hat{w}_t + \Omega_{\varphi} \varphi_{t+1} + \frac{1}{2} \hat{w}_t' \Omega_{ww} \hat{w}_t + \frac{1}{2} \varphi_{t+1} \Omega_{\varphi\varphi} \varphi_{t+1} + \hat{w}_t' \Omega_{w\varphi} \varphi_{t+1}, \tag{67}
\]

\(^{37}\)Vector with all zero elements except one denoting the interest rate.

\(^{38}\)For detailed description of kernel derivation please refer to Appendix C.
where matrices $\Omega_w$, $\Omega_\varphi$, $\Omega_{ww}$, $\Omega_{\varphi\varphi}$ and $\Omega_{w\varphi}$ consists of structural parameters of the macro model.\footnote{Again, since the derivation is not straightforward, please see Appendix C for detailed description.}

Comparing the last two equations “financial solution” in (66) and “macro solution” in (67) one can notice that the form is the same, so we can easily assign macro parameters (in $\Omega$’s) to kernel characteristics (price of risk $\Lambda$’s) and thus express the yield curve in terms of macroeconomic variables. To emphasize that, a crucial point is to ensure that deep parameters of the macro model (mentioned in Table 6) having reasonable values allows $\Lambda_0$ and $\Lambda_1$ be functions of respective $\Omega$’s, so equations (66) and (67) equal. Later, this will allow for providing simulations of macroeconomic shocks on term structure of interest rate.

### 3.3.2 Price of bonds

Having derived pricing kernel, I have to specify prices of bonds ($Q^n_t$), which in literature usually take form

$$Q^n_t = \exp(A_n + B_n' w_t), \quad (68)$$

or in a more tractable logarithmic deviations from the steady state

$$\hat{q}_t^n = A_n + B_n' \hat{w}_t. \quad (69)$$

$\hat{w}_t$ denotes vector of macroeconomic variables from macro solution 65, $A_n$ and $B_n$ are level and slope parameters of yield curve, since $n$-period yield ($r^n_t$) is nothing else then

$$r^n_t = -\frac{\hat{q}_t^n}{n} = \frac{A_n + B_n' \hat{w}_t}{n}. \quad (70)$$

### Yield curve parameters

In order to get a consistent model of the yield curve, level and slope parameters must be also connected with macro model solution. Parameters are solved recursively simply by substituting to the equation that defines price of bonds (54). It is useful to work with its logarithmic version

$$\hat{q}_t^n = \mathbb{E}_t(\hat{m}_t + \hat{q}_{t+1}^{n-1}) + \frac{1}{2}V_t(\hat{m}_{t+1} + \hat{q}_{t+1}^{n-1}), \quad (71)$$

where $V_t$ denotes the function of variation (errors) in price of bonds. Substituting for kernel ($\hat{m}_t$) from equation (67) and for price of bonds ($\hat{q}_{t+1}^{n-1}$) from (69),\footnote{Substitution is done by shifting (69) by one period, so $\hat{q}_{t+1}^{n-1} = A_{n-1} + B_{n-1}' \hat{w}_{t+1}$.} we can derive desired parameters

$$A_n = A_{n-1} + \frac{1}{2}B_{n-1}' \Theta \Sigma \Theta' B_{n-1} - B_{n-1}' \Theta \Sigma \Lambda_0, \quad (72a)$$

$$B_n' = B_{n-1}' (\Phi - \Theta \Sigma \Lambda_1) - e_3'. \quad (72b)$$
where \( \Sigma \) is matrix denoting respective error terms from macro variables.\(^{41}\) Initial values are for \( A_1 = 0 \) and \( B_1 = e_3 \) being a selection vector for the policy rate. One period yield thus corresponds with the short term interest rate such as \( r^1_t = i_t \).

### 3.3.3 Complete macro-finance model

Finally, the full model can be expressed by two equations

\[
\begin{align*}
\dot{w}_t &= \Phi \dot{w}_{t-1} + \Theta \varphi_t, \\
\hat{R}_t &= A + B \hat{w}_t + \eta_t,
\end{align*}
\]

(73a) (73b)

where first one is the solution of macro model (65);\(^{42}\) \( \hat{R}_t = (\hat{r}^1_t, \hat{r}^4_t, \hat{r}^{12}_t, \hat{r}^{20}_t, \hat{r}^{28}_t, \hat{r}^{40}_t) \), \( \eta_t = (\eta^1_t, \eta^4_t, \eta^{12}_t, \eta^{20}_t, \eta^{28}_t, \eta^{40}_t)' \). Matrices \( \Phi, \Theta, A \) and \( B \) are reduced form matrices of deep parameters of the macro-finance model.

#### Term premia

Having derived complete macro-finance model, we have all information about term structure of interest rates (\( \hat{R}_t \)). This allows to derive a term premia \( (\zeta^p_t) \), which is defined as a difference between \( n \)-period bond rate and expected future short term rates. It is one of factors explaining long term yields.

\[
\zeta^p_t = r^n_t - \frac{1}{n} \sum_{j=0}^{n-1} E_t(i_{t+j}).
\]

(74)

Substituting from complete model (73) we can rewrite the equation as

\[
\zeta^p_t = \frac{1}{n} \sum_{j=0}^{n-1} \left\{ B'_j \Theta \Sigma \Lambda_0 - 0.5 B'_j \Theta \Sigma \Theta' B_j \right\} + \left[ B'_j (I - \Phi + \Theta \Sigma \Lambda_1) + e'_j (I - \Phi^j) \right] \dot{w}_t.
\]

(75)

The term premia represents a measure of departure from pure expectation hypothesis.\(^{43}\) As Bernanke \textit{et al.} (2004) mention, “...long term yields are determined by two components: (1) the expected future path of one-period interest rate and (2) the excess returns that investors demand as a compensation for the risk of holding longer-term instruments”.

---

\(^{41}\) \( \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). For detailed derivation please refer to Appendix D.

\(^{42}\) Thus \( \dot{w}_t = (\dot{y}_t, \dot{y}_{t-1}) \), \( \varphi_t = (\epsilon_t, \epsilon'_o) \).

\(^{43}\) Term premia is calculated for respective price of an asset (here for government bonds), so it does not consider riskiness of an asset.
4 Time series approach

As discussed previously, macro-finance modelling often relies on econometric VAR models. Typically they include two or three explanatory macro variables (such as future interest rates, expected inflation, past inflation, unemployment gap, etc.) and three so called latent (unobservable) variables. We also establish this type of model for a comparison with my structural DSGE approach developed in the previous Chapter 3.

4.1 VAR specification

The benchmark model is inspired by a very popular application of micro-finance modelling of Ang & Piazzesi (2003), often cited in literature. However, it is not the only representative model, also e.g. Cochrane & Piazzesi (2002), Kim & Orphanides (2005) or Joslin et al. (2011) should be mentioned among others. In order to obtain comparable results we use the same macroeconomic variables as in the case of DSGE approach, i.e. private consumption, government expenditures, interest rate and inflation. Variables are assumed to follow VAR(2) process, so the number of lags are equal to those from DSGE as mentioned in equation (64). The VAR(2) model has a standard form

\[ \begin{align*}
    x_o^t &= \Phi_1 x_o^{t-1} + \Phi_2 x_o^{t-2} + \Theta_1 \epsilon_o^t, \\
    (76)
\end{align*} \]

where \( x_o^t = (c_t, g_t, i_t, \pi_t)' \) is \( (4 \times 1) \) vector of macro variables, \( \Theta_1 \) is \( (4 \times 4) \) lower triangular variance-covariance matrix of errors and \( \epsilon_o^t = (\epsilon_c^t, \epsilon_g^t, \epsilon_i^t, \epsilon_{\pi}^t)' \) is a vector of respective variables’ estimation errors with \( \epsilon_o^t \sim N(0, I), iid. \) It is convenient to rewrite model into VAR(1) specification\(^{45}\)

\[ \begin{align*}
    \begin{bmatrix}
        x_o^t \\
        x_o^{t-1}
    \end{bmatrix} &= \begin{bmatrix}
        \Phi_1 & \Phi_2 \\
        I & 0
    \end{bmatrix} \begin{bmatrix}
        x_o^{t-1} \\
        x_o^{t-2}
    \end{bmatrix} + \begin{bmatrix}
        \Theta_1 & 0 \\
        0 & 0
    \end{bmatrix} \begin{bmatrix}
        \epsilon_o^t \\
        0
    \end{bmatrix}, \\
    (77)
\end{align*} \]

Unobservable (latent) factors added to the model, relates to level \( (x_u^{u1}) \), slope \( (x_u^{u2}) \) and curvature \( (x_u^{u3}) \). Three latent factors have been identified as sufficient to explain the yield dynamic by various studies, such as Knez et al. (1994) or more recently by analysis of Pericoli & Taboga (2008). Factors are assumed to follow simple AR(1) process

\[ \begin{align*}
    x_u^t &= \Phi_u x_u^{t-1} + \Theta_u \epsilon_u^t, \\
    (78)
\end{align*} \]

where \( x_u^t = (x_u^{u1}, x_u^{u2}, x_u^{u3}) \), \( \Phi_u \) is \( (3 \times 3) \) lower triangular matrix, \( \Theta_u = I \) assigns shocks with factors and \( \epsilon_u^t \) is vector of errors of unobservable components.

\(^{44}\)To make things easier for a reader, we are using the same notation in both models where possible. But to be clear, none of results or parameters come from DSGE model here. In other words, parameters may be noted the same, but their value and/or estimation technique differs between models.

\(^{45}\)Detailed discussion about VAR transformations and analysis can be found in Hamilton (1994).
Combination of the last two equations for observable and unobservable factors leads to the form of final model

\[
\begin{bmatrix}
    x^o_t \\
    x^o_{t-1} \\
    x^u_t
\end{bmatrix}
= \begin{bmatrix}
    \Phi_1 & \Phi_2 & 0 \\
    I & 0 & 0 \\
    0 & 0 & \Phi^u
\end{bmatrix}
\begin{bmatrix}
    x^o_{t-1} \\
    x^o_{t-2} \\
    x^u_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
    \Theta_1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & \Theta^u
\end{bmatrix}
\begin{bmatrix}
    \varepsilon^o_t \\
    \varepsilon^o_{t-1} \\
    \varepsilon^u_t
\end{bmatrix}.
\]

In the model, there is an independence between macro and latent factors, which is apparent from matrices of parameters having zero elements in their last rows and columns attributed to macro variables.\(^{46}\) The result, can be written in an abbreviated form\(^{47}\)

\[
w_t = \Phi w_{t-1} + \Theta \varphi_t.
\]

Short rate dynamics is defined as an affine structure of factors

\[
i_t = \delta_0 + \delta_1^t w_t,
\]

where the short term rate is assumed to depend on current values of macro factors, so vector \(\delta_1\) is (11 x 1) and has all elements zero, except the first four.

**Pricing kernel**

Similarly to the DSGE model, pricing kernel is derived under assumption of no arbitrage, which implies that a single pricing kernel can price all assets

\[
M_{t+1} = \exp(-i_t - 0.5\Lambda'_t \Lambda_t - \Lambda'_t \varepsilon_{t+1} + \eta_{t+1}),
\]

with time varying market price of risk defined in terms of factors

\[
\Lambda_t = \Lambda_0 + \Lambda_1 w_t.
\]

where \(\Lambda_0\) is (11 x 1) a vector determining long term mean of yields and \(\Lambda_1\) (11 x 11) matrix identifying how shocks in state variables affect all yields - i.e. affects time-variation of term premia. Here we assume that \(\Lambda_t\) depends on contemporary macro variables and latent factors only.

**Term structure of interest rates**

Interest rate term structure is an affine function of state variables\(^{48}\) as in equation 70, i.e.

\[
R_t = r_t^n = \frac{A_n + B'_n w_t}{n},
\]

\(^{46}\)Imposed independence between latent and macro factors assigns upper-right 8 x 3 corner and lower left 3 x 8 corner to be zeros.

\(^{47}\)Vector \(w_t\) is of (11 x 1) dimension, since \(x^o_t\) and \(x^o_{t-1}\) are both (4 x 1) and \(x^u_t\) is (3 x 1). Partitioned matrices \(\Phi\) and \(\Theta\) consists of respective square block matrices.

\(^{48}\)Latent factors included in vector state variables can be derived from \(R_t\) and parameters \(A\) and \(B\) by inversion from equation

\[
R_t = \frac{A_n + B'_n x^o_t + B''_n x^u_t}{n}.
\]
for all maturities included in the model $R_t = (r^1_t, r^4_t, r^{12}_t, r^{20}_t, r^{28}_t, r^{40}_t)$. Parameters $A_n$ and $B_n$ take the same form as in equation (72) (or as derived in Appendix D), i.e.

$$A_n = A_{n-1} + \frac{1}{2} B'_{n-1} \Theta \Sigma \Theta' B_{n-1} - B'_{n-1} \Theta \Sigma \Lambda_0 - \delta_0,$$  

$$B_n' = B_{n-1}' (\Phi - \Theta \Sigma \Lambda_1) - \delta_1,'$$

with initial values for $A_1 = -\delta_0$ and $B_1 = -\delta_1$.\(^{49}\)

### 4.2 Estimation

Estimation of such model is not very straightforward. First, it is important to define macro dynamics by estimating parameters $\Phi_1$, $\Phi_2$ and $\Theta_1$ in equation (76), which can be easily done using least square estimate (OLS). This approach can be also used in case of interest rate dynamics in equation (81) to find values for $\delta_0$ and $\delta_1$.

The problematic part is to estimate parameters $A_n$ and $B_n$ (or price of risk parameters $\Lambda_1$, $\Lambda_0$ respectively); in other words to solve equations (85). Ang & Piazzesi (2003) estimate $\Lambda$s using Maximum Likelihood (ML) estimation of equation (84). But as they also admit, these estimates are not easy to obtain, because likelihood function is usually very flat, which complicates identification of global optimum.\(^{50}\)

It has been shown by Joslin et al. (2011), Bauer et al. (2011) and Hamilton & Wu (2012) among others, that the estimation process can be simplified to large extent, using simple OLS estimates. This perfectly works for just identified models, but can also be applied to those with overidentified restrictions when followed by numerical minimization of quadratic difference between estimated reduced form parameters and values that are implied by model parameters ($\Phi_1$, $\Phi_2$ and $\Theta_1$). This procedure is showed to be asymptotically equivalent to ML estimates and applicable for this class of models.

When estimating VAR model parameters we will follow the latter application proposed by Hamilton & Wu (2012), in which an interested reader may also find an application to Ang & Piazzesi (2003) model.\(^{51}\)

We need to describe yields of six maturities $R_t = (r^1_t, r^4_t, r^{12}_t, r^{20}_t, r^{28}_t, r^{40}_t)$, as in previous model. Using the Hamilton-Wu approach, we separate the yields into two groups: first group is measured without an error ($R^1_t$) and second one, which is priced with an error $r^1_t = \delta_0 = 0$ and $\delta_1$ is a choosing vector, having all elements zero except one assign to short term interest rate.

\(^{49}\)For $n=1$, i.e. short term rate $r^1_t = i_t$ while $\delta_0 = 0$ and $\delta_1$ is a choosing vector, having all elements zero except one assign to short term interest rate.

\(^{50}\)This problem is mentioned also in other studies, e.g. Kim & Orphanides (2005) or Hamilton & Wu (2012).

\(^{51}\)Hamilton and Wu made data and their very useful MATLAB code for estimations of Affine Term Structure Models available online at http://dss.ucsd.edu/~jhamilto/software.htm.
The first group is usually associated with unobservable latent factors. Generally, the system of equations to estimate is as follows:

\[
\begin{bmatrix}
  x_t^o \\
  R_1^t \\
  R_2^t
\end{bmatrix} =
\begin{bmatrix}
  A_1 \\
  A_2
\end{bmatrix}
+ \begin{bmatrix}
  0 & \Phi_1 & \Phi_2 & 0 \\
  B_1^0 & B_1^1 & B_2^0 & B_2^1
\end{bmatrix}
\cdot
\begin{bmatrix}
  x_{t-1}^o \\
  x_{t-2}^o \\
  x_t^o
\end{bmatrix}
+ \begin{bmatrix}
  \Theta_1 & 0 & 0 \\
  0 & B_1^0 & B_1^1
\end{bmatrix}
\cdot
\begin{bmatrix}
  \epsilon_t^o \\
  \epsilon_t^u
\end{bmatrix}, \quad (86)
\]

Where desired parameters \( A_n \) and \( B_n \) are

\[
A_n = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B_n = \begin{bmatrix} B_1^0 & B_1^1 & B_2^0 & B_2^1 \\ B_1^0 & B_2^0 & B_2^1 & B_2^2 \end{bmatrix}. \quad (87)
\]

Minimization\(^{52}\) is then focused on minimizing diagonal matrix \( \Theta^r \).

### 4.3 Term premia

Having estimated all necessary parameters of the macro-finance model and describing yields, term premia (defined as difference between \( n \)-period bond rate and expected future short term rates) takes the same form as in previous Chapter, i.e.

\[
\zeta_t^p = \frac{1}{n} \sum_{j=0}^{n-1} \left\{ \left[ B_j^I \Theta \Sigma \Lambda_0 - 0.5 B_j^I \Theta \Sigma \Theta^r B_j \right] + \right.
\]

\[
+ \left[ B_j^I (I - \Phi + \Theta \Sigma \Lambda_1) + \epsilon_j^I (I - \Phi^j) \right] \hat{w}_t \right\}. \quad (88)
\]

\(^{52}\)For general details about the method see Rothenberg (1971). Application to Affine Term Structure models can be found in mentioned Hamilton & Wu (2012).
5 Data and results

Having derived both models, the structural DSGE and the second one based on VAR with latent (unobservable) factors, we can compare the two models and illustrates basic transmission mechanisms.

Firstly, we show their ability to fit an average yield curve observed from the data and then I analyze impacts of shocks to different variables using impulse response analysis (IRF). Results of both models may differ and they do, which is mainly due to construction of DSGE model that gives more detailed structure of the economy than VAR model. So while some effects are clearly visible in results of VAR, they may be offset by some opposite effect so the response in DSGE is very mild. Or on the other hand, effects may go in the same direction and DSGE response is more pronounced.

5.1 Data

The analysis uses data for the US, which benefits from very long time series starting usually in late 40s (or in late 60s in some cases). I use seasonally adjusted quarterly data for the period beginning from 1Q1970 and ending in 4Q2015, so there are 184 observations giving quite strong data background for estimations.\textsuperscript{53}

Bond yields include maturities of 1, 4, 12, 20, 28 and 40 quarters Treasury constant maturity rates. Macroeconomic variables represents private and government consumption, 3-months short term interest rate and inflation (CPI) in terms of growth rates.\textsuperscript{54}

\textsuperscript{53}Good sources of data for US economy are those of Federal Reserve Bank of St. Louis (http://research.stlouisfed.org/fred2/) mainly for macroeconomic data and of Bureau of Economic Analysis (http://www.bea.gov/) for fiscal data.

\textsuperscript{54}Interest rate is per quarter here.
Figure 5: Data used

(a) Bond Yields

(b) Macro Factors

Table 1 presents characteristics of the yield data. Yields are rising until beginning of 80s, then they are declining to the current lowest levels. Shape of an average yield curve is upward sloping and increases are smaller with maturity. Standard deviation is decreasing with higher maturities. Series are quite persistent with autocorrelation higher that 0.9 (measured by ACF(1) coefficient).
Table 1: Descriptive statistics of US Government Bond Yields

<table>
<thead>
<tr>
<th>Statistics</th>
<th>3 months</th>
<th>1 year</th>
<th>3 years</th>
<th>5 years</th>
<th>7 years</th>
<th>10 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>5.49</td>
<td>5.99</td>
<td>6.47</td>
<td>6.75</td>
<td>6.97</td>
<td>7.10</td>
</tr>
<tr>
<td>std. deviation</td>
<td>3.18</td>
<td>3.34</td>
<td>3.13</td>
<td>2.95</td>
<td>2.82</td>
<td>2.70</td>
</tr>
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<td>skewness</td>
<td>0.49</td>
<td>0.49</td>
<td>0.44</td>
<td>0.53</td>
<td>0.61</td>
<td>0.71</td>
</tr>
<tr>
<td>kurtosis</td>
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<td>3.40</td>
<td>3.27</td>
<td>3.27</td>
<td>3.24</td>
<td>3.25</td>
</tr>
<tr>
<td>autocorrelation</td>
<td>0.956</td>
<td>0.959</td>
<td>0.964</td>
<td>0.965</td>
<td>0.968</td>
<td>0.970</td>
</tr>
</tbody>
</table>

Source: Federal Reserve Bank of St. Louis., own calculations.

Data include the effect of economic crisis, especially in second half of 2008 and in 2009, however it does not seem to be substantial problem since impacts on estimations were affected only to very limited extent.\textsuperscript{55}

5.2 Yield curve estimates

Applying both models to the data and providing necessary estimates we are able to show some results of models’ performance.

Figure 6 shows ability of models to fit the average yield curve, where real data are depicted by black dots and represents means for each maturity over the data sample. Both models can do very well in estimating and fitting the data and differences are very small, even for 10-year rates.

\textsuperscript{55}Due to this fact, we do not publish results of sensitivity analysis on crisis here.
The yield curve, on one side of equation, is linear in parameters $A_n$, $B_n$ that characterize level (intercept) and slope of the yield curve as shown in equations (73b) or (84). Graphical illustration of maturity dependent parameters (Figure 7) for both model shows that intercept increases with maturity. Intercept $A_n$ do not differ much for all maturities, since both models fit the level of yield curve accurately. Slope $B_n$, also derived from structural parameters of each model, is shown on pictures on the right side. It shows how individual underlying factors affect $B_n$ for each maturity (on axes $x$). Not surprisingly, the interest rate is the dominant factor in both models, affecting all maturities with declining magnitude.

Figure 7: Maturity dependent parameters $A_n$, $B_n$ for individual factors

![Graphical illustration of maturity dependent parameters](image)

(a) DSGE

(b) VAR

Other factors contribute to the yield curve development to lower extent. VAR indicates government spending to be more persistent, through overall higher estimated parameter in VAR. DSGE shows inflation and consumption as more dominant factors. Techno-
logical innovations are highly persistent factors and thus they affect yield curve mainly from medium term maturities to its long-end.

Term premia is defined as a difference between long term rates and expected (projected) future short term rates. Both models gives quite similar estimates, especially in the last two decades, with quite visible impact of recent financial crisis. This negative term premia can be viewed as a price for uncertainty about future development of interest rates.

Figure 8: Term premia

![Plot of Term Premia](image)

Table 2: Descriptive statistics of term premia

<table>
<thead>
<tr>
<th>model</th>
<th>mean</th>
<th>std.deviation</th>
<th>skewness</th>
<th>kurtosis</th>
<th>normality(^1)</th>
<th>autocorrelation(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSGE</td>
<td>0.40</td>
<td>0.21</td>
<td>-0.61</td>
<td>4.86</td>
<td>0.00</td>
<td>0.73</td>
</tr>
<tr>
<td>VAR</td>
<td>0.68</td>
<td>0.25</td>
<td>0.33</td>
<td>4.04</td>
<td>0.01</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Note: \(^1\)p-value of JB test, \(^2\)ACF(1) coefficient.

There are usually non-negligible differences in estimates of the term premia between models. To show differences I use an illustration taken from the paper of Rudebusch et al. (2007) that compares five different approaches.\(^{56}\) When comparing these results

\(^{56}\)Individual estimates are results from following studies (beside the one cited): Bernanke et al. (2004), Cochrane & Piazzesi (2008), Kim & Wright (2005) and Rudebusch & Wu (2008).
with our models, none of them is an outlier and they both lie within a reasonable range given by published estimates.

Figure 9: Different estimates of term premia

Source: Rudebusch et al. (2007).
Note: Graph is for 10-year term premia, denoting difference between 10-year maturity as the longest in the model and sum of short term rates.

5.3 Macroeconomic impacts

Another way how to think about models and their results is to test how they respond to various economic shocks. We introduce here four basic shocks, to private and government consumption, interest rate and inflation (resp. technology), i.e. to variables included in the model. They enter the model through respective exogenous shocks $\delta$ in case of DSGE and by shocks $\epsilon$ in VAR.\footnote{For reference see the equation (49) in Section 3.2.2 for DSGE and the equation (76) for VAR.} Thus preference shock for consumption enters through $\delta^c$ in the first model (using $\epsilon^c$ in the second model), government consumption shock through $\delta^g$ ($\epsilon^g$ in VAR) and so on.

Performance of each model is illustrated by Impulse Response Functions IRF - percentage deviations from steady state of each variable over a period of time. In many cases results are the same since IRFs lie within or close to the interval. Some results are quite similar when IRFs have the same shape but they differ in magnitude. Or finally, they even differ in sign. The last two cases can be mainly explained by difference in nature
of DSGE\textsuperscript{58} and sometimes problematic estimates of VAR.  

Since we are using the simple VAR model for the purpose of comparison with the structural approach, the same number of two lags are introduced in both models. This form is very standard for the DSGE.\textsuperscript{59} Differences in results in case of VAR could be, to some extent, overcome by more careful choice of lags in variables.\textsuperscript{60} Another interesting approach would be to introduce Sign Restricted VAR instead of this simple VAR, where the benchmark would be the structural model.\textsuperscript{61}

Figure 10: Preference shock

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\textsuperscript{58}It should be kept in mind, that different impacts of shocks in models may come from their different introduction to the model. While in DSGE shocks are exogenous variables following AR(1) process in VAR they are imposed by error terms.  

\textsuperscript{59}Usually DSGE models are then rewritten to AR(1) form as in equation (53). For the purposes of derivation of financial part of the model, we extended the number of lags in equations (64) and (65).  

\textsuperscript{60}E.g. as in case of Ang & Piazzesi (2003), where authors use 12 lags of two explanatory macro variables.  

\textsuperscript{61}This approach tests and corrects signs of responses to different shocks. The paper does not aim at detailed analysis of VAR models, so we do not elaborate on this in detail. Discussion of this approach can be found in e.g. Canova & Paustian (2007).
First Figure 10 shows impacts of consumption innovations. Responses look quite similar and results of structural model lie close to confidence bands. Positive preference shock is reflected in higher private consumption as well as government consumption that can be afforded thanks to higher budgetary income. Higher demand induces higher production and increase in prices.\footnote{Mainly due to higher firms’ costs that are pushed by increasing wages. Although wages are not explicitly modelled, this transmission channel is present in the model.} Higher production activity and an increase in prices induce higher interest rates as can be inferred from the monetary policy reaction equation (47).

Both models differ a little bit in terms of magnitude of IRFs and also in persistence. Preference shock has very similar impact on consumption itself. Effects on other variables seem to be a bit overestimated in terms of VAR model. Mainly it is the case of interest rate, that reverts to its steady state long time after consumption shock.

Figure 11: Government expenditure shock

Positive shock to government consumption (Figure 11) brings up some discrepancies. Comparing to preference shock in DSGE, government shock has a smaller impact on
production due to its lower weight in total GDP (20% comparing to 80% of private consumption). However, the higher demand from government induce slightly higher prices (again through firms’ costs) followed by an increase in interest rates. Effect on private consumption is very small here and is induced by higher interest rates and higher prices. In other words, there is no direct crowding out effect caused by higher government demand. Having this story from the structural model, VAR shows opposite effects on interest rate and prices that stems from estimated negative coefficients between these variables.

Figure 12: Interest rate shock

A shock to the short term interest rate (Figure 12) also brings some diverse results. As in case of previous government shock, VAR restrictions or modified numbers of lags would be probably beneficiary here too. On the other hand, the DSGE makes the story quite clear. The shock to the short term rate is viewed by consumers as a signal to spend less on their current consumption, which is thus deferred to the future. Lower demand and lower production does not allow the government to keep the level of spending that also decreases. Due to these reasons inflation slows down.
The difference between results is in persistency of these impacts too, which is notable also in case of the preference shock. DSGE converges more quickly – up to ten quarters, which seems to be a reasonable amount of time for adjusting of real economy to one-off shock in short term interest rates.

Last shock is imposed on the fourth explanatory macroeconomic variable, prices. Figure 13 is slightly different from those above. In case of DSGE the fourth shock would be a technology one, which affects firms’ production process through decreasing costs (in case of positive shock). However, it is not possible to make similar shock to VAR model due to lack of this transmission channel, so we can stick to inflation shock only. Obviously these two shocks would have an opposite impact on real variables (positive technology pushes prices down, while positive inflation shock increases them directly). So for the purpose of this comparison we introduce a negative technology shock in the case of DSGE.

As results show, positive shock to prices (induced by negative technology shock) leads to decrease in private and government consumption that become more costly. Inflation
pressures then induce a reaction of monetary authority and short term interest rate increases. There are two offsetting factors affecting the short term rate. Lower demand results in decrease in interest rates, but higher inflation, more important for monetary policy, does the opposite. Technology shock, representing a structural change in an economy, is also more pronounced and more persistent than a simple shock to prices.

Results revealed that interpretation of VAR may be sometimes little bit problematic. It relies on statistical approach, while structural model uses fundamental relations between variables to explain developments in the economy. However, this analysis uses very simple VAR model with two lags, so using proposed methods at the beginning of this Chapter may lead to an improvement of the results.

5.4 Yield curve impacts

Applying the same shocks, it is also possible to see impacts on yield curve through IRFs of short and long term yields as shown in Figure 14. First column shows responses of a short term one-year yield (four quarters \( r^4 \)) and second column includes results for long term 7-years yield (\( r^{28} \)).

Impacts on yields are in terms of decimal percentage points and the impact is lower with the higher maturity. Persistence of shocks tends to be the same across maturities. As in the case of macroeconomic impacts, the VAR exhibit more persistent and pronounced results. An exception is the technology shock that is, as in case of macro results, naturally longer-lasting than inflation shock in case of VAR.

Having shown the comparison of basic VAR(2) model with DSGE approach, both models give comparable results despite some differences. Structural approach is thus able to fit the yield curve data and to give answers about impacts of shocks to the economy and also to the yield curve. Moreover, DSGE models give us more information about the structure of the economy and transmission channels between real variables. These models are also already widely used for macroeconomic analysis, so the financial part can be possibly incorporated. However, this has also its limits since the derivations and estimations of parameters would be more painful for large scale models. On the other hand, larger model with more explanatory variables would probably provide us with more degrees of freedom for estimations, which might contribute to the model stability.
Figure 14: Yield curve impacts

\[ \delta^{c,c} \rightarrow r^4 \]

\[ \delta^{g,g} \rightarrow r^4 \]

\[ \delta^{i,i} \rightarrow r^4 \]

\[ \delta^{z,\pi} \rightarrow r^4 \]

quarters after shock

\[ \delta^{c,c} \rightarrow r^{28} \]

\[ \delta^{g,g} \rightarrow r^{28} \]

\[ \delta^{i,i} \rightarrow r^{28} \]

\[ \delta^{z,\pi} \rightarrow r^{28} \]

quarters after shock

DSGE

VAR
6 Implications

Previous Chapters pointed out possible implications for real economic policies. Current economic analysis widely use the structural approach\(^{63}\) for modelling and forecasting of main macroeconomic variables, such as GDP, inflation, unemployment, net exports, short term interest rates etc. However, these analysis usually neglect impacts on long term rates. This part shows two simple examples when further analysis of various macroeconomic impacts may be important. First case is monetary policy, which plays crucial role in stabilizing price level and make decisions about responding to economic situation based on results of their models. Secondly, this issue is important also for debt management institutions, mainly Treasuries, Ministries of Finance and Economy.

6.1 Monetary policy

Central banks (CEB) decide about main interest rate based on economic analysis and forecasts carried out by structural models.\(^{64}\) In this process it is quite important to distinguish between sources of movements (shocks) in the economy and whether these changes are permanent or rather temporary. In other words, whether the CEB should respond to certain shocks or not. Unfortunately the channel to long term rates is usually missing in general models and Central Banks assess the financial markets impacts quite separately from the real economy analysis.

In the real world, they are long term rates that matters in many aspects. There are several examples, starting with the case of households – consumers’ loans usually have longer maturities, so the short term interest rate (typically 3-month interest rate) is not sufficient to explain households’ behaviour. Also firms decide about their investment projects into fixed capital for many years ahead. And finally governments keep certain structure of their debt (in terms of maturities) and decide about bond emissions. So CEB must be aware of impacts on term structure of interest rates and needs to consistently assess the real economy with financial markets.

For illustration, lets consider different situations – different shocks into the economy that model allows to analyze. Graphs of impulse responses are not very different from those in previous Chapter. All lines represent positive unit shocks to consumption preference, government spending, short term interest rate and technology. Impacts of each of these four shocks are depicted on separate graphs in Figure 15. Showing all maturities together allows for a clear illustration of the pass-through to the yield structure. Two main findings can be inferred from charts. Each shock has the largest impact on short term yields so the magnitude declines with maturity. And each shock has different

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\(^{63}\)This Chapter focuses on results from the DSGE based model only.

\(^{64}\)Examples among others are e.g. Bank of England, Sveriges riksbank, Czech National Bank, Banque de France, Central Bank of Japan.
For example a positive unit shock to the consumption (consumption preference) represented by the top left chart translates into approx. 0.05p.p increase in 1-year yield, 0.02p.p in 3-years yield and 0.005 in long 7-years yield. These effects vanish in approximately ten quarters after shock. The same applies to the positive shock to government spending and to short term rate (e.g. change in CEB rate), with the difference of shock duration. The effect on yields relates to impacts of these shocks on macroeconomic variables. The previous Chapter showed that all three shocks have positive impacts on short term rates, which is also transformed into yields. The positive technology shock, definitely the most persistent one, on the other hand drives interest rates down and makes resulting effect more persistent.

Figure 15: Implication of shocks on yield term structure

In case of interest rate shock is the largest impact on short end of the yield curve – mostly on 3-months and 1-year maturities. However, the pass-through to longer maturities is still present.\textsuperscript{66} By changing the policy rate, CEBs affect rates that are important for the economy.

\textsuperscript{65}By lowering firms’ costs and slows down inflation.

\textsuperscript{66}Although it may not be apparent from the lower left graph due to the scale.
Because Central Banks aim at price stability, inflation development plays an important role in decision about changing a monetary policy rate. Figure 16 shows estimated impacts of all (positive) shocks on inflation. Having in mind that possible intervention affects also medium term interest rates, it should be considered to what kinds of shocks to respond. Concerning the four illustrative shocks\textsuperscript{67} it might not be necessary to take an action in case of government and interest rate shocks. Their impacts are either mild and/or not very long-lasting. Consumption shock with higher persistence is then a better candidate to induce a policy action. Finally the technology shock, without any doubt, is the one which CEB should respond to, since it diverts inflation grow from its target for a long period of time.

![Figure 16: Implications of shocks on inflation](image)

\textbf{6.2 Fiscal policy}

One of important roles of fiscal authorities (Treasuries) is to provide debt management and to decide about its structure and exposure to different maturities. Having the US as an example case, the total amount of outstanding federal debt in 2015 reached 18.9 trillion USD, from which more than 68% is held by public (of which only 5% is non-marketable), the rest is in Federal Government Accounts.

\textsuperscript{67} And fact that they are recognized by analysis to perform like on the graph. Here we also assume that CEB aims at inflation grow to be zero (i.e. constant inflation rate).
The Table shows a composition of US debt over past several years. Absolute majority of the debt (90%) has maturity up to 10 years, i.e. horizon that is covered in this analysis. A great part is distributed into short and medium term maturities, while in recent years there is visible tendency to switch into bonds with rather longer maturities. This relates to programs\textsuperscript{68} introduced as a reaction to economic crisis aiming at lowering medium and longer term interest rates in order to support an economic growth.

Set of graphs in Figure 17 compares the baseline and an alternative scenario for each type of shock. The first line for the preference shock shows the two scenarios of cost of debt payments (in billions USD) between 2013 and 2015. First chart depicts the difference for bonds up to one year maturity, second for 1–5 years and third for 5–10 years maturities. Last one is for total effect across all maturities. The same applies for other lines that are assigned to government spending, interest rate and technology shock.

\textsuperscript{68}More specifically: Large Scale Asset Purchase programs (LSAP) and Maturity Exchange Program (MEP).

\textsuperscript{69}In this case we simply assume the real data to be a baseline and responses to shocks are added to baseline in order to show the difference in interest payments stemming from changes in term structure of interest rates other things being equal. We are aware that this is very simplified calculation, which does not consider other factors such as e.g. demand for bonds. We analyze the part of the debt with maturity up to 10 years only which is the longest maturity covered by the model. Numbers does not have to comply with official statistics since we use simplified approach and do not distinguish more complicated structure of the debt.
Differences expressed in quarterly costs show the magnitude of impacts and persistence of each shock. Let’s take the upper left chart, showing the impact of preference shock on costs of debt denominated in short maturities, as an example. From previous IRF function (first graph in Figure 15) for this shock it is visible that its impact on yields lasts for approximately ten quarters. Translated to this illustration, the costs of debt are likely to be higher from first quarter of 2013 (when the shock is materialized) until third quarter of 2015 when the effect vanishes.

Not surprisingly, results show biggest differences in case of short term maturities, since they are affected to the largest extent. Technology shock being the most persistent one and influencing also long maturities has the biggest impact on debt costs. Apparently,
due to very low short term interest rates, this alternative scenario hits zero lower bound so the effect is not fully materialized here.

To sum up these effects for specific maturities, Figure 18 show differences between scenarios in respective years taking into account the maturity distribution of the debt. An increase in the short term rate induces costs mainly in the first year after the shock stemming from short term maturities only. As this shock has a lower impact on longer maturities, it would be less costly to have more debt denominated in longer term bonds.

Figure 18: Impacts of shocks to different maturities

The same applies also to preference shock, which has a smaller impact but is more persistent and thus raises costs in two consecutive years. Technological shock to the structure of the economy (that has a positive impact) on the other hand lowers costs for longer period of time and beside short maturities it also affects longer ones non-negligibly.

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70Assuming other things equal, so we do not consider any other offsetting measures, which would have possibly taken place.
71It is worth noting that the impact of technology shock to short yields up to 1–year maturity is somewhat lower due to hitting zero lower bound. We do not assume negative costs of the debt. This
To get better illustration of impacts on total costs induced by each shock in respective years, I sum up over maturities for each type of shock. Results in Table 4 are additional costs (savings) of the debt induced by particular macroeconomic shock through affecting the term structure of yields.\textsuperscript{72}

Table 4: Change in interest payments due to shocks (nominal, in bn USD)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>2013</th>
<th>2014</th>
<th>2015</th>
</tr>
</thead>
<tbody>
<tr>
<td>preference shock</td>
<td>14.66</td>
<td>9.19</td>
<td>1.31</td>
</tr>
<tr>
<td>gov. spending shock</td>
<td>1.66</td>
<td>1.46</td>
<td>0.71</td>
</tr>
<tr>
<td>interest rate shock</td>
<td>20.39</td>
<td>-2.40</td>
<td>-1.01</td>
</tr>
<tr>
<td>technology shock</td>
<td>-96.54</td>
<td>-88.43</td>
<td>-83.12</td>
</tr>
</tbody>
</table>

Source: Own calculations.

Persistence of shocks matters and it should be considered whether to make some changes in portfolio or not. In case of relatively mild shock that diminishes quickly, the implementation might take even longer time and thus it is probably not worthy to react. But when changes in preferences or in technology are recognized, some adjustment in maturity structure probably should be considered. Table 5 brings an useful information in this sense, showing shares of debt maturing in following months.

Table 5: Percentage of marketable debt maturing in next 12 to 36 months

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12-Months</td>
<td>41</td>
<td>39</td>
<td>30</td>
<td>28</td>
<td>27</td>
<td>25</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>24-Months</td>
<td>55</td>
<td>50</td>
<td>44</td>
<td>41</td>
<td>40</td>
<td>39</td>
<td>38</td>
<td>39</td>
</tr>
<tr>
<td>36-Months</td>
<td>60</td>
<td>59</td>
<td>54</td>
<td>51</td>
<td>50</td>
<td>49</td>
<td>49</td>
<td>49</td>
</tr>
</tbody>
</table>

Note: End of period data.

There are usually about one third of bonds that mature in next 12 months and more than 50% that do so in next 3 years. This number is slightly decreasing mainly due to the switch to longer term bonds.

is obvious from the first graph in last line in Figure 17.
\textsuperscript{72}All other things being equal.
7 Conclusion

Even though general equilibrium framework may be subject to a criticism, the truth is that it offers an integrated view of economy as a whole. DSGE modelling aims at explaining behaviour of the real economy. So the focus is on e.g. GDP and its components, inflation, wages, unemployment and interest rate among others. The lastly mentioned interest rate is in these models typically approximated by a short term rate as a proxy of Central Bank’s policy rate. However it serves well for this purpose, the real economy mostly depends on medium and long term interest rates (households’ mortgages, firms’ fixed investments, etc.). But medium and long term rates are missing in these macro models.

Connecting DSGE models with models of yield term structure is beneficial for both sides. DSGE offers a good framework to explain dynamics of underlying macroeconomic variables for the yield curve and conversely structural models need to find a way how to incorporate long term rates.

DSGE model is able to overcome some of VAR’s drawbacks. VAR models based on purely statistical approach are sometimes difficult to interpret. One should carefully analyze characteristics of underlying time series and decide about how many lags to include. VAR models are also sometimes “corrected” in their sign (Sign Restricted VAR), using structural approach as a benchmark, in order to follow economic reasoning. In terms of macro-finance models, their explanatory macro variables are supplied also by three unobservable (latent) factors that explains level, slope and curvature of the yield curve. When estimating pricing kernel, restrictions are imposed on these factors in order not to allow for their mutual interactions. In other words there is no impact of macro variables on latent factors and vice versa. And finally, problems may occur when estimating the price of risk, using Maximum Likelihood Function, which is quite challenging for finding a global optimum.

DSGE can overcome these problems thanks to their already mentioned characteristics. Some of them are quasi structural with some restrictions or with time invariant term premia. The latter is a handicap for explaining yield curve dynamics. Some other models resulted in non-affine (nonlinear) structure, which is hardly tractable and computational burden prevents them (at least recently) from being used widely.

Therefore we focus on a structural macro-finance model, that is clear in explaining main links in the economy and answering questions about impacts of shocks. Even though one should always carefully decide how to analyze and determine exogenous shocks in order to accurately simulate a phenomenon of interest. The model is also able to fit an average yield curve observed in the data. It is derived consistently with basics of term premia is one of two main components (beside expected future short term rates) that explains long term yields.
financial approach, i.e. we are able to derive price of risk and pricing kernel using macro variables and structural parameters only. It is a big advantage of this approach because it translates the dynamics from the structure of macro model to financial part. On the other hand, in case of VAR, price of risk is an estimated parameter. Even though the derived kernel is in DSGE version highly nonlinear, its approximation by second order Taylor expansion results in deriving time varying term premia.

Comparison of results from the two models shows that both of them are doing well in fitting yield curve data and their resulting term premia is in line with published estimates. Results of VAR are more difficult to interpret. Also more attention should be paid to the choice of appropriate number of lags included as some results seem to be quite overestimated. Results also revealed that what latent factors explain in VAR models, DSGE is able to identify directly by macro variables. Thus slope can be explain predominantly by interest rate (or monetary policy) shock and level by technology shock.

However DSGE has also its limits, mainly for large sized models. Already in case of this four-equation model, the derivations behind are little bit time consuming and may become cumbersome or even hardly manageable in case of large all-embracing models. Problems may also arise with adding some specific variables to the model, e.g. applying this “extension” to an open economy DSGE model may bring an issue of capital flows and exchange rate movements that would complicate situation.

We also demonstrated an importance of macro-finance analysis for monetary and fiscal authority using simple examples. In the first case the analysis confirms that the Central Bank, through changes in its policy rate, influences also medium and partly long term rates beside short ones. Since these rates are important for real economy CEB should be aware of this when deciding about responding to some shocks. Mainly it depends on magnitude and persistence of an impact that the respective shock has on inflation, which is the most important criterion for monetary authority. E.g. government consumption shock or interest rate shock are rather moderate or not long-lasting, monetary policy does not have to necessarily take an action in order to avoid an impact on interest rate term structure. On the other hand in case of technology shock, which is quite pronounced and very persistent, accommodation might be advisable. All shocks that were introduced seem to follow a similar pattern, having the largest impact on short term rates that decreases with higher maturities.

Case study of fiscal policy illustrates how costs of debt, may be affected by shocks to consumption preference, government spending, interest rate and inflation, showing an importance of this issue for debt management. The analysis stem from the total amount of US debt and its maturity distribution. Four shocks applied to analyze how would costs of debt have had been developing under alternative scenarios. Not surprisingly the biggest effect has a technology shock affecting costs for a long time, while others
have more moderate impact for a period up to two years. Even though it is very simple example, it reveals a useful information for considering maturity distribution of the debt.

Even though the presented DSGE model is rather simple and collapses into four equations, it does not necessarily mean that results are somehow biased. There is no perfect correlation between adding more variables (equations) and added value to the results. It is obvious from derivations that more variables increase the size of the model quite largely and thus make its tractability less easy since derivations are becoming more unpleasant.
References


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Appendix A: DSGE model

Households

Households maximize utility function

$$\max_{\{C_t, N_t, B_t\}} U_\tau = E_\tau \sum_{t=\tau}^{\infty} \beta^t \delta_t^c \left[ \frac{(C_t - H_t^c)^{1-\psi_c}}{1-\psi_c} - \frac{(N_t)^{1+\psi_n}}{1+\psi_n} \right],$$  \hspace{1cm} (89)$$

subject to budget constraint

$$Q_t B_t + (1 + \tau^p) P_t C_t = Q_{t-1} B_{t-1} + (1 - \tau^w + \tau^b) W_t N_t + (1 - \tau^f) \Pi_t,$$  \hspace{1cm} (90)$$

By derivations we get FOC

$$\frac{\partial L}{\partial C_t} : \beta^t \zeta (C_t - H_t)^{-\psi_c} = \lambda_t (1 + \tau^p) P_t,$$  \hspace{1cm} (91a)$$

$$\frac{\partial L}{\partial N_t} : \beta^t \zeta N_t^{\psi_n} = \lambda_t (1 - \tau^w + \tau^b) W_t,$$  \hspace{1cm} (91b)$$

$$\frac{\partial L}{\partial B_t} : - \lambda_t \beta^t Q_t = \lambda_{t+1} \beta^{t+1} Q_{t+1}.$$  \hspace{1cm} (91c)$$

Substituting 91c to 91b for both $$\lambda$$s we get FOC for consumption

$$1 = \beta^t E_t \left[ \frac{(C_{t+1} - H_{t+1}^c)}{C_t - H_t^c} \right]^{-\psi_c} \frac{1 + i_t}{1 + \pi_{t+1}} \frac{\delta_{t+1}^c}{\delta_t^c}.$$  \hspace{1cm} (92)$$

Putting together 91c and 91b results in FOC for labour supply

$$N_t^{\psi_n} = \frac{(1 - \tau^w + \tau^b) W_t}{(1 + \tau^p) P_t} (C_t - H_t^c)^{-\psi_c}.$$  \hspace{1cm} (93)$$

Firms

In order to derive a demand for individual good, firms focus on minimization of cost function by choosing amount of production of individual goods in a basket

$$\min_{\{Y_{it}\}} P_t Y_t = \int_0^1 P_t Y_{it} d_i,$$  \hspace{1cm} (94)$$

subject to an aggregate output demand

$$Y_t = \left[ \int_0^1 (Y_{it})^{\frac{\sigma - 1}{\sigma}} d_j \right]^{\frac{\sigma}{\sigma - 1}},$$  \hspace{1cm} (95)$$

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with first order condition
\[
\frac{\partial Y_t}{\partial Y_{it}} = \frac{\sigma}{\sigma - 1} \left[ \int_0^1 (Y_{it})^{\frac{\sigma - 1}{\sigma}} \, di \right]^{-\frac{1}{\sigma - 1}} \frac{\sigma - 1}{\sigma} (Y_{it})^{-\frac{1}{\sigma}}. 
\] (96)

In optimum, the value of marginal product equals to price
\[
\frac{Y_{it}^{-\frac{1}{\sigma}}}{Y_t^{-\frac{1}{\sigma}}} = \frac{P_{it}}{P_t} \quad \Rightarrow \quad \frac{Y_{it}}{Y_t} = \left( \frac{P_{it}}{P_t} \right)^{-\sigma}. 
\] (97)

We can substitute this to 94 and after mathematical operations derive price index
\[
P_t = \left[ \int_0^1 (P_{it})^{1-\sigma} \, di \right]^{\frac{1}{1-\sigma}}. 
\] (98)

Resulting demand for individual goods equals to
\[
Y_{it} = Y_t \left( \frac{P_{it}}{P_t} \right)^{-\sigma}. 
\] (99)

**Price setting**

There are firms that reset their prices and those who keep them from previous period
\[
P_t = \left[ \int_0^1 (P_{it})^{1-\sigma} \, di \right]^{\frac{1}{1-\sigma}}. 
\] (100)

Those who reset prices in a particular period do that in two ways
\[
P_{it}^* = \chi P_{it}^b + (1 - \chi) P_{it}^f, 
\] (101)
i.e. either by backward looking rule
\[
P_{it}^b = (1 + \pi_{it-1}) P_{it-1}^*, 
\] (102)
or by forward looking optimization. This firm does so by solving optimization problem
\[
\max_{(P_{it})} \Pi^P_{tr} = \mathbb{E}_r \sum_{t=r}^{\infty} (\beta \xi)^t (P_{it} - MC_t) Y_{it}, 
\] (103)
subject to the demand for individual good
\[
Y_{it} = Y_t \left( \frac{P_{it}}{P_t} \right)^{-\sigma}, 
\] (104)
The result is as follows
\[
\frac{\partial \Pi^P_{it}}{\partial P_{it}} = \mathbb{E}_t \sum_{t=0}^{\infty} (\beta \xi)^t \left[ (1 - \sigma) P_{it}^{-\sigma} Y_t \sigma - MC_t Y_t (-\sigma) P_{it}^{-\sigma-1} P_{it}^\sigma \right] = 0, \tag{105}
\]
or rearranged as
\[
\frac{\partial \Pi^P_{it}}{\partial P_{it}} = \mathbb{E}_t \sum_{t=0}^{\infty} (\beta \xi)^t \left[ (1 - \sigma) \left( \frac{P_{it}^{-\sigma}}{P_{it}^\sigma} \right) Y_t + \sigma MC_t Y_t \left( \frac{P_{it}^{-\sigma}}{P_{it}^\sigma} \right) \frac{1}{P_{it}} \right] = 0. \tag{106}
\]
Marginal costs are then derived from costs that a firm can possibly have (the only costs here are those of labour), i.e.
\[
\max_{\{N_t\}} \Pi^P_{it} = \mathbb{E}_t \sum_{t=\tau}^{\infty} (\beta \xi)^t (P_{it} - MC_t) Y_{it}, \tag{107}
\]
subject to
\[
Y_{it} = \delta^z_t (N_t)^\alpha, \tag{108}
\]
resulting in
\[
\frac{\partial \Pi^P_{it}}{\partial N_t} = P_{it} \delta^z_t \alpha N_t^{\alpha-1} - W_t = 0. \tag{109}
\]
After mathematical operations,\footnote{Including substitution from production function for \(\delta^z_t = \frac{Y_{it}}{N_t^\alpha}\).} I get
\[
\frac{W_t}{P_{it}} = \alpha \frac{Y_{it}}{N_t}. \tag{111}
\]

**Fiscal policy**

Fiscal policy operates with revenues
\[
GR_t = \tau^w W_t N_t + \tau^f \Pi_t + \tau^p P_t Y_t + \delta^R_t \tag{112}
\]
and expenditures
\[
GE_t = G_t P_t + \tau^b W_t N_t + \delta^e_t. \tag{113}
\]
Fiscal policy rule aims at stabilizing budget \(D = GE_t - GR_t = 0\). From this an “optimal” consumption in real terms is
\[
G^o_t = \tau^w W_t N_t + \tau^f \Pi_t + \tau^p Y_t + \delta^R_t - \tau^b W_t N_t + \delta^e_t, \tag{114}
\]
rearranged as

$$G_t^a = (\tau^w - \tau^b) \frac{W_t N_t}{P_t} + \tau^f \frac{P_t Y_t - W_t N_t}{P_t} + \tau^p Y_t + \delta_t^g. \quad (115)$$

Final version of government consumption equals to

$$G_t = (1 - \phi_g) \left( \tau^f \frac{P_t Y_t - W_t N_t}{P_t} + \tau^p Y_t + \delta_t^g \right) + \phi_g \frac{G_{t-1}}{P_{t-1}}. \quad (116)$$
Appendix B: Parameters of DSGE model

Consumption equation

\[ \omega_{cf} = \frac{1}{1 + \gamma}, \]  
\[ \omega_{cl} = \frac{\gamma}{1 + \gamma}, \]  
\[ \omega_{ci} = \frac{1 - \gamma}{\psi_c(1 + \gamma)}, \]  
\[ \omega_{cz} = \frac{(1 - \gamma)(1 - \rho_c)}{\psi_c(1 + \gamma)}. \]  

Government consumption equation

\[ \omega_{gc} = \frac{(1 - \phi_g) \left[ \mu_g (1 - \sigma) \frac{\psi_c}{1 - \gamma} + \omega_{yc} \frac{(1 + \psi_n - \alpha)}{\alpha} + \omega_{yc} \right]}{1 - (1 - \phi_g) \omega_{yg} \left[ \mu_g (1 - \sigma) \frac{1 + \psi_n - \alpha}{\alpha} + 1 \right]}, \]  
\[ \omega_{gcl} = \frac{(1 - \phi_g) \mu_g (1 - \sigma) \frac{\psi_c}{1 - \gamma}}{1 - (1 - \phi_g) \omega_{yg} \left[ \mu_g (1 - \sigma) \frac{1 + \psi_n - \alpha}{\alpha} + 1 \right]}, \]  
\[ \omega_{ggl} = \frac{\phi_g}{1 - (1 - \phi_g) \omega_{yg} \left[ \mu_g (1 - \sigma) \frac{1 + \psi_n - \alpha}{\alpha} + 1 \right]}, \]  
\[ \omega_{gzg} = \frac{(1 - \phi_g)(1 - \sigma) \mu_g \frac{1 + \psi_n}{\alpha}}{1 - (1 - \phi_g) \omega_{yg} \left[ \mu_g (1 - \sigma) \frac{1 + \psi_n - \alpha}{\alpha} + 1 \right]}, \]  
\[ \omega_{gzz} = \frac{(1 - \phi_g)(1 - \sigma) \mu_g \frac{1 + \psi_n}{\alpha}}{1 - (1 - \phi_g) \omega_{yg} \left[ \mu_g (1 - \sigma) \frac{1 + \psi_n - \alpha}{\alpha} + 1 \right]}. \]  

Interest rate equation

\[ \omega_{ic} = \omega_{yc} \lambda_y (1 - \phi_i), \]  
\[ \omega_{ig} = \omega_{yg} \lambda_y (1 - \phi_i), \]  
\[ \omega_{ip} = \lambda_p (1 - \phi_i), \]  
\[ \omega_{ii} = \phi_i, \]  
\[ \omega_{iz} = 1 - \phi_i. \]
Inflation equation

\[ \omega_{ppl} = \frac{\chi}{\xi + \chi[1 - \xi(1 - \beta)]}, \quad (120a) \]
\[ \omega_{ppl} = \frac{\beta \xi}{\xi + \chi[1 - \xi(1 - \beta)]}, \quad (120b) \]
\[ \omega_{pc} = \frac{(1 - \chi)(1 - \xi)(1 - \beta \xi)}{\xi + \chi[1 - \xi(1 - \beta)]} \left[ \frac{\psi_c}{1 - \gamma} + \omega_{yc} \frac{1 + \psi_n - \alpha}{\alpha} \right], \quad (120c) \]
\[ \omega_{pc} = \frac{(1 - \chi)(1 - \xi)(1 - \beta \xi)}{\xi + \chi[1 - \xi(1 - \beta)]} \left[ \frac{\psi_c \gamma}{1 - \gamma} \right], \quad (120d) \]
\[ \omega_{pg} = \frac{(1 - \chi)(1 - \xi)(1 - \beta \xi)}{\xi + \chi[1 - \xi(1 - \beta)]} \left[ \omega_{yg} \frac{1 + \psi_n - \alpha}{\alpha} \right], \quad (120e) \]
\[ \omega_{pz} = \frac{(1 - \chi)(1 - \xi)(1 - \beta \xi)}{\xi + \chi[1 - \xi(1 - \beta)]} \frac{1 + \psi_n}{\alpha}. \quad (120f) \]
## Parameter values

Table 6: Parameters of the model

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<tr>
<th>parameter</th>
<th>description</th>
<th>value</th>
<th>method</th>
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<tr>
<td>( \beta )</td>
<td>discount factor (“impatience”)</td>
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<td>( \psi_n )</td>
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<tr>
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<td>calculated</td>
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<tr>
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Appendix C: Pricing kernel

We stem from pricing kernel in 58 equal to

\[ M_{t+1} = \beta E_t \left[ \left( \frac{C_{t+1} - H^c_{t+1}}{C_t - H^c_t} \right)^{-\psi_c} \frac{1}{1 + \pi_{t+1} \delta^c_{t+1}} \right] , \]  

expressed in logarithms of variables as

\[ M_{t+1} = \exp(- (r_t + \pi_{t+1}) + (\delta^c_{t+1} - \delta^c_t) - \psi_c \ln(c_{t+1} - \gamma c_t) + \psi_c \ln(c_t - \gamma c_{t-1})) . \]  

We can transform that to a matrix form

\[ M_{t+1} = \exp \left( \begin{bmatrix} \hat{c}_{t+1} \\ \hat{\pi}_{t+1} \\ \hat{\delta}_{t+1} \\ \hat{\delta}_t \\ \hat{\delta}_{t-1} \end{bmatrix} \right) , \]

and modify for next calculations

\[ M_{t+1} = \exp \left( \begin{bmatrix} \hat{c}_{t+1} \\ \hat{\pi}_{t+1} \\ \hat{\delta}_{t+1} \\ \hat{\delta}_t \\ \hat{\delta}_{t-1} \end{bmatrix} \right) , \]

where \( k_t = (\hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{\delta}_{t+1})' \), \( k_{2t} = (\hat{c}_t, \hat{\delta}_t)' \) and \( k_{3t} = \hat{\delta}_{t-1} \). Vector of all variables is then \( k = (k_{1t}, k_{2t}, k_{3t})' \).

We approximate kernel by the second order Taylor expansion

\[ M_{t+1} \approx \bar{M}_{t+1} + \bar{g}'(k_t - \bar{k}_t) + \frac{1}{2}(k_t - \bar{k}_t)'\bar{H}(k_t - \bar{k}_t) , \]  

or in a shorter form of deviations

\[ \hat{m}_{t+1} = \bar{g}' \hat{k}_t + \frac{1}{2} \hat{k}' \bar{H} \hat{k}_t , \]

where \( \hat{m}_{t+1} = (M_{t+1} - \bar{M}_{t+1})/(\bar{M}_{t+1}) \) and gradient includes partial derivatives of each \( k \), i.e. \( g = (g_1', g_2', g_3') \). In full, it takes the following form

\[ \hat{m}_{t+1} = \begin{bmatrix} g_1' & g_2' & g_3' \end{bmatrix} \begin{bmatrix} \hat{k}_{1t} \\ \hat{k}_{2t} \\ \hat{k}_{3t} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \hat{k}_{1t} & \hat{k}_{2t} & \hat{k}_{3t} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} \hat{k}_{1t} \\ \hat{k}_{2t} \\ \hat{k}_{3t} \end{bmatrix} . \]

We can apply this solution to macroeconomic variables in \( \hat{k} \)s derived from macro model \( \hat{y}_t \) (see equation 53). However, not all variables from that vector are used for pricing
kernel, so it is necessary to introduce choosing matrices \((D_1, D_2, D_3)\), which denote (pick) desired variable from general vector \(\hat{y}_t = (\hat{c}_t, \hat{g}_t, \hat{\pi}_t, \hat{\delta}_t, \hat{\delta}_t, \hat{\delta}_t)'\). Thus

\[
\hat{k}_{1t} = (\hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{\delta}_{t+1})' = D_1\hat{y}_{t+1} = \begin{bmatrix} e_1' \\ e_4' \\ e_5' \end{bmatrix} \hat{y}_{t+1}, \quad (125a)
\]

\[
\hat{k}_{2t} = (\hat{c}_t, \hat{\delta}_t)' = D_2\hat{y}_t = \begin{bmatrix} e_1' \\ e_5' \end{bmatrix} \hat{y}_t, \quad (125b)
\]

\[
\hat{k}_{3t} = (\hat{c}_{t-1})' = D_3\hat{y}_{t-1} = [e_1'] \hat{y}_{t-1}. \quad (125c)
\]

Vectors \(e\) has all elements equal to zero except that indicated by index. Those equals to one. After appropriate mathematical operations\(^{75}\) We end up with quite cumbersome form

\[
\hat{m}_{t+1} = (g'_1 D_1 \Phi_1 + g'_2 D_2)\hat{y}_{t-1} + (g'_3 D_3)\hat{y}'_{t-1} + g'_1 D_1 \Theta_1 \epsilon_{t+1} +
\]

\[
+ \frac{1}{2} \begin{bmatrix} \hat{y}'_t (D'_1 \Phi'_1 H_{11} D_1 \Phi_1 + D'_2 H_{21} D_1 \Phi_1 + D'_1 \Phi'_1 H_{12} D_2 + D'_2 H_{22} D_2) \hat{y}_t +
\hat{y}'_{t-1} (D'_3 H_{31} D_1 \Phi_1 + D'_3 H_{32} D_2) \hat{y}_t +
\hat{y}'_{t-1} (D'_1 \Phi'_1 H_{13} D_3 + D'_2 H_{23} D_3) \hat{y}_{t-1} +
\hat{y}'_{t-1} (D'_3 H_{33} D_3) \hat{y}_{t-1} \end{bmatrix} +
\]

\[
+ \frac{1}{2} \epsilon_{t+1} (D'_1 \Theta'_1 H_{11} D_1 \Theta_1) \epsilon_{t+1} +
\]

\[
+ \frac{1}{2} \epsilon_{t+1} (D'_1 \Theta'_1 H_{11} D_1 \Theta_1) \epsilon_{t+1} +
\]

\[
+ \frac{1}{2} \epsilon_{t+1} (D'_1 \Theta'_1 H_{11} D_1 \Theta_1) \epsilon_{t+1}, \quad (126)
\]

that can be simplified using the useful extension from equations 63 – 65 and thus can be the final form as in equation 67 derived:

\[
\hat{m}_{t+1} = \Omega_w \hat{w}_t + \Omega_{\varphi} \varphi_{t+1} + \frac{1}{2} \hat{w}'_t \Omega_{ww} \hat{w}_t + \frac{1}{2} \varphi'_{t+1} \Omega_{\varphi \varphi} \varphi_{t+1} + \hat{w}'_t \Omega_{w \varphi} \varphi_{t+1}. \quad (127)
\]

Crucial point for consistent macro-finance model is to is to ensure that this macroeconomic interpretation of the kernel is consistent with the financial specification

\[
\hat{m}_{t+1} = - (\Lambda'_0 A_1 + e_3') \hat{w}_t - \Lambda'_0 \varphi_{t+1} - \frac{1}{2} \hat{w}_t \Lambda'_1 A_1 \hat{w}_t - \frac{1}{2} \Lambda'_0 \Lambda_0 - \hat{w}_t \Lambda'_1 \varphi_{t+1}, \quad (128)
\]

\(^{75}\)And substitutions for future values with \(\hat{y}_{t+1} = \Phi_1 \hat{y}_t + \Theta_1 \epsilon_{t+1}\) from equation 53.
so that respective parameters equal, i.e. mainly

\[ \Lambda_0 = -\Omega_\varphi, \]  
\[ \Lambda_0 = \Omega_{w\varphi}, \]  
\[ \Lambda'_0 \Lambda_1 + e'_3 = \Omega_w. \]  

(129a)  
(129b)  
(129c)
Appendix D: Yield curve parameters

Yield curve parameters $A_n, B_n$ are derived from pricing kernel, price of risk and final equation of macro model.

\[ \hat{m}_{t+1} = -\hat{i}_t - 0.5\Lambda'_t A_t - \Lambda'_t \varphi_{t+1}, \quad (130a) \]
\[ \Lambda_t = \Lambda_0 + \Lambda_1 \hat{w}_t, \quad (130b) \]
\[ \hat{w}_t = \Phi \hat{w}_{t-1} + \Theta \varphi_t. \quad (130c) \]

In affine models, price of bonds is given by

\[ \hat{q}_t^n = A_n + B'_n \hat{w}_t, \quad (131) \]

expressed in logarithm

\[ \hat{q}_t^n = \mathbb{E}_t[\hat{m}_{t+1} + \hat{q}_{t+1}^{n-1}] + 0.5 V_t(\hat{m}_{t+1} + \hat{q}_{t+1}^{n-1}), \quad (132) \]

with $V_t$ being conditional volatility term.

Elements in last equations equal

\[ \mathbb{E}_t[\hat{m}_{t+1}] = -\hat{i}_t - 0.5\Lambda'_t A_t, \quad (133a) \]
\[ \mathbb{E}_t[\hat{q}_{t+1}^{n-1}] = A_{n-1} + B'_{n-1} \mathbb{E}_t[\hat{w}_{t+1}], \]
\[ = A_{n-1} + B'_{n-1} \Phi \hat{w}_t. \quad (133b) \]

\[ V_t(\hat{m}_{t+1} + \hat{q}_{t+1}^{n-1}) = V_t(-\Lambda'_t \varphi_{t+1} + A_{n-1} + B'_{n-1} \mathbb{E}_t[\hat{w}_{t+1}]), \]
\[ = V_t(-\Lambda'_t \varphi_{t+1} + B'_{n-1} \Theta \varphi_t), \]
\[ = (B'_{n-1} \Theta - \Lambda'_t) V_t \varphi_{t+1} (B'_{n-1} \Theta - \Lambda'_t)', \]
\[ = B'_{n-1} \Theta \Sigma \Theta' B_{n-1} + \Lambda' \Sigma A_t - 2B'_{n-1} \Theta \Sigma A_t. \quad (133c) \]

while in the last equation $\Sigma = \varphi \varphi'$. Substituting back to the bond price equation 132 we get

\[ \hat{q}_t^n = -\hat{i}_t - 0.5\Lambda'_t A_t + A_{n-1} + B'_{n-1} \Phi \hat{w}_t \]
\[ + 0.5 \left[ B'_{n-1} \Theta \Sigma \Theta' B_{n-1} + \Lambda' \Sigma A_t - 2B'_{n-1} \Theta \Sigma A_t \right], \]
\[ = A_{n-1} + 0.5B'_{n-1} \Theta \Sigma \Theta' B_{n-1} - B'_{n-1} \Theta \Sigma A_0 \]
\[ + B'_{n-1} \Phi \hat{w}_t - e_3' \hat{w}_t + B'_{n-1} \Theta \Sigma A_1 \hat{w}_t \]
\[ = A_n + B'_n \hat{w}_t. \quad (134a) \]

where we substituted for $\Lambda_t = \Lambda_0 + \Lambda_1 \hat{w}_t$ and $e_3' \hat{w}_t = \hat{i}_t$. Finally, parameters dependent on $\hat{w}_t$ determine $B_n$ and those remaining $A_n$, such as

\[ A_n = A_{n-1} + 0.5B'_{n-1} \Theta \Sigma \Theta' B_{n-1} - B'_{n-1} \Theta \Sigma A_0, \quad (135a) \]
\[ B'_n = B'_{n-1}(\Phi - \Theta \Sigma A_1) - e'_3. \quad (135b) \]
Appendix E: Term premia

Term premia is defined as follows

\[ \zeta_p^t = r^t_n - \frac{1}{n} \sum_{j=0}^{n-1} E_t(i_{t+j}). \tag{136} \]

After substitution for yield \( r^t_n = \frac{q^t_n}{n} \) I get

\[ \zeta_p^t = \frac{1}{n} \left[ -q^t_n - \sum_{j=0}^{n-1} E_t[i_{t+j}] \right] = \frac{1}{n} \sum_{j=0}^{n-1} \left( q^t_j - q^t_{j+1} - i_{t+j} \right), \tag{137} \]

where \( q^t_j - q^t_{j+1} \) can be denoted as forward rate premia. Elements in the last equation equal

\[ E_t[i_{t+j}] = e_3' E_t \hat{w}_{t+j} = e_3' \Phi^j \hat{w}_t, \tag{138a} \]

\[ q^t_j - q^t_{j+1} = (A_j + B_j' \hat{w}_t) - (A_{j+1} + B_{j+1}' \hat{w}_t) = [B'_j \Theta \Sigma \Lambda_0 - 0.5 B'_j \Theta \Sigma \Theta' B_j] + [B'_j (I - \Phi + \Theta \Sigma \Lambda_1) + e_3'] \hat{w}_t, \tag{138b} \]

which comes from substitution for parameters \( (A_j - A_{j+1}) \) and \( (B_j - B_{j+1}) \) derived in Appendix 7. Substituting back to equation 137 we get final the form

\[ \zeta_p^t = \frac{1}{n} \sum_{j=0}^{n-1} \left\{ [B'_j \Theta \Sigma \Lambda_0 - 0.5 B'_j \Theta \Sigma \Theta' B_j] + [B'_j (I - \Phi + \Theta \Sigma \Lambda_1) + e_3' (I - \Phi^j)] \hat{w}_t \right\}. \tag{139} \]